1. A toll bridge charges $1.00 for passenger cars and $2.50 for other vehicles. Suppose that 60% of all vehicles are passenger cars. If 25 vehicles cross the bridge during a particular daytime period,

(a) what is the probability that at least 3 of them are passenger cars? Use both formula and R-function to compute this.

(b) what is the expected revenue and its associated standard deviation? (Note that the revenue is a linear function of the number of passenger cars)

2. Each of 12 refrigerators of a certain type has been returned to a distributor because of an audible, high-picked, oscillating noise. Suppose that 7 of these refrigerators have a defective compressor and cost $100 each to repair, and 5 of them have less serious problems, and cost $50 each to repair. If the refrigerators are examined in a random order, let $X$ be the number of refrigerators with defective compressors among the first 6 examined ones.

(a) Calculate $P(X = 4)$ and $P(X \leq 4)$. Use both formula and R-function to compute this.

(b) What is the expected cost of repairing these 6 refrigerators and its associated SD?

3. Prove the following results.

(a) If $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$, and $X$ and $Y$ are independent, show that $X+Y \sim Poisson(\lambda+\mu)$. Hint: Use the binomial theorem $\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a^i b^{n-i} = (a+b)^n$.

(b) If $X \sim Unif(0,1)$, show that $Y = -\log(X) \sim Exponential(1)$. Hint: compute cdf $F(y) = P(Y \leq y)$.

4. Vehicle speed $V$ on a particular bridge follows a normal distribution.

(a) If the mean and standard deviations are respectively 65 m/h and 5 m/h approximately how many percents of vehicles have speed between 60 m/h and 75 m/h?

(b) If top 2% are to be ticketed, what is the minimum speed to be ticketed?

(c) If 5% of all vehicles travel less than 50 m/h and 10% travel more than 70 m/h, what are the mean and standard deviation of vehicle speed?

5. Consider again adjusted closing prices of Apple Inc. and Johnson & Johnson from Jan. 1, 2000 to September 8, 2016. Use $qqnorm$, $qqplot$ to answer the following questions.

(a) Do the returns of Apple Inc., and Johnson & Johnson follow a normal distribution?

(b) Compare the tails of the returns of Apple Inc and Johnson & Johnson with a $t$-distribution with degree of freedom 4.
(c) Compare the distributions of the returns of Johnson & Johnson during the 2008 financial crisis (index: 2063:1812, from 7/1/08 – 6/30/09) with those two years after financial crisis (index: 1306:1, from 7/1/11–9/8/16).

(d) What is the appropriate degree of freedom of $t$-distribution for modeling the returns of the Apple stock two years after financial crisis (index: 1306:1, from 7/1/11–9/8/16).

6. Let $S_t$ denote the price of a stock with current value $S_0 = $100. Suppose that its daily log return $r_t = \log(S_t/S_{t-1})$ follows independent with equal chance to gain/lose 1%: $P(r_t = 0.01) = P(r_t = -0.01) = 0.5$. Then, it is easy to see $S_t = 100 \exp(r_1 + \cdots + r_t)$. \(^1\)

(a) Use the R-function `cumsum` to simulate a price path $\{S_t\}_{t=0}^{252}$ (no for loop is allowed for this problem). Give a time series plot of your simulated path.

(b) Simulate 1000 paths to calculate the expected stock price and its associated standard deviation in one year\(^2\), i.e., $E(S_{252})$ and $SD(S_{252})$. What is the probability that the stock price is higher than 100? Please use the loop operation for 1000 simulations.

(c) Loop is slow in R. The following matrix operation eliminates the loop.

\[
\begin{align*}
X &= \text{matrix}(rbinom(252000, 1, 0.5), nrow=1000, ncol=252) \\
&\quad \# create a 1000x252 random matrix with Bernoulli(0.5) \\
X &= 2*X -1 \quad \# convert to -1 and 1. \\
Y &= \text{apply}(X, 1, \text{sum}) \quad \# adding up and down days for each path \\
Z &= 100*\exp(0.01*Y) \quad \# 1000 simulated stock prices in one year
\end{align*}
\]

Please complete the remaining code and answer questions in part (b).

You are welcome to compare the speed with 10000 paths or double loops (through days and simulations), but this is not required for the homework.

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\(^1\)This is an approximation to the “Geometric Brown motion”.

\(^2\)There are approximately 252 trading days every year.