Dynamic Asset Allocation: 
a Portfolio Decomposition Formula 
and Applications

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1 Introduction

- Dynamic consumption-portfolio choice:
  - Merton (1971): optimal portfolio includes intertemporal hedging terms in addition to mean-variance component (diffusion)
  - Breeden (1979): hedging performed by holding funds giving best protection against fluctuations in state variable (diffusion)
  - Ocone and Karatzas (1991): representation of hedging terms using Malliavin derivatives (Ito, complete markets)
    → Interest rate hedge
    → Market price of risk hedge
  - Detemple, Garcia and Rindisbacher (DGR JF, 2003): practical implementation of model (diffusion, complete markets)
    → Based on Monte Carlo Simulation
    → Flexible method: arbitrary number of assets and state variables, non-linear dynamics, arbitrary utility functions
    → Extends to incomplete/frictional markets (DR MF, 2005)
Contribution:

- **New decomposition of optimal portfolio (hedging terms):**
  - Formula rests on change of numéraire: use pure discount bonds as units of account
  - Passage to a new probability measure: forward measure (Geman (1989) and Jamshidian (1989))
  - General context: Ito price processes, general utilities
New economic insights about structure of hedges:

→ Utility from terminal wealth: hedge
  · fluctuations in instantaneous price of long term bond with maturity date matching investment horizon
  · fluctuations in future bond return volatilities and future market prices of risk (forward density)
  · first hedge has a static flavor (static hedge)

→ Utility from terminal wealth and intermediate consumption
  · static hedge is a coupon-paying bond, with variable coupon payments tailored to consumption needs

→ Risk aversion properties:
  · if risk aversion approaches one both hedges vanish: myopia
  · if risk aversion becomes large mean-variance term and second hedge vanish: holds just long term bonds
  · if risk aversion vanishes all terms are of first order in risk tolerance.

→ Non-Markovian $N + 2$ fund separation theorem.
• Technical contribution:
  → Exponential version of Clark-Haussmann-Ocone formula
  → Identifies volatilities of exponential martingale in terms of Malliavin derivatives
  → Malliavin derivatives of functional SDEs
  → Explicit solution of a Backward Volterra Integral Equation (BVIE) involving Malliavin derivatives.
Applications:

- Preferred habitat
- Extreme risk aversion behavior
- International asset allocation
- Preferences for I-bonds
- Integration of risk management and asset allocation

Road map:

- Model with utility from terminal wealth
- The Ocone-Karatzas formula
- New representation
- Intermediate consumption
- Applications
- Conclusions
2 The Model

- Standard Continuous Time Model:
  - Complete markets and Ito price processes
  - Brownian motion $W$, $d$-dimensional
  - Flow of information $\mathcal{F}_t = \sigma(W_s : s \in [0, t])$
  - Finite time period $[0, T]$.
  - Possibly non-Markovian dynamics
Assets: Price Evolution

- **Risky assets (dividend-paying assets):**

  \[
  \frac{dS_t^i}{S_t^i} = (r_t - \delta_t^i) \, dt + \sigma_t^i \left( \theta_t \, dt + dW_t \right), \quad S_0^i \text{ given}
  \]

  - \( \sigma_t^i \): volatility coefficients of return process \((1 \times d \text{ vector})\)
  - \( r_t \): instantaneous rate of interest
  - \( \delta_t^i \): dividend yield
  - \( \theta_t \): market prices of risk associated with \( W \) \((d \times 1 \text{ vector})\)
  - \( (r, \delta, \sigma, \theta) \): progressively measurable processes; standard integrability conditions

- **Riskless asset:**

  - pays interest at rate \( r \)
• **Portfolio policy** $\pi$: $d$-dimensional, progressively measurable; integrability conditions

\[ \rightarrow \text{amounts invested in assets: } \pi \]

\[ \rightarrow \text{amount in money market: } X - \pi' \mathbf{1} \]

• **Wealth process:**

\[
\begin{aligned}
    dX_t &= r_t X_t dt + \pi'_t \sigma_t (\theta_t dt + dW_t), \\
    &\text{subject to } X_0 = x.
\end{aligned}
\]

• **Admissibility:** $\pi$ is admissible ($\pi \in \mathcal{A}$) if and only if wealth is non-negative: $X \geq 0$. 
Asset Allocation Problem:

- **Investor maximizes expected utility of terminal wealth:**
  \[
  \max_{\pi \in A} \mathbb{E}[U(X_T)]
  \]

- **Utility function:** \( U : \mathbb{R}_+ \to \mathbb{R} \)
  - Strictly increasing, strictly concave and differentiable
  - Inada conditions: \( \lim_{X \to \infty} U'(X) = 0 \) and \( \lim_{X \to 0} U'(X) = \infty \)
- Includes **CRRA** \( U(x) = \frac{1}{1-R} X^{1-R} \) where \( R > 0 \).

- **Property:**
  - Strictly decreasing marginal utility in \((0, \infty)\)
  - Inverse marginal utility \( I(y) \) exists and satisfies \( U'(I(y)) = y \)
  - Derivative: \( I'(y) = 1/U''(I(y)) \)
3 The Optimal Portfolio

Complete Markets:

- **Market price of risk:** $\theta_t = (\theta_{1t}, \ldots, \theta_{dt})'$

- **State price density:**
  \[
  \xi_v \equiv \exp \left( - \int_0^v (r_s + \frac{1}{2} \theta'_s \theta_s) \, ds - \int_0^v \theta'_s \, dW_s \right)
  \]

  → converts state-contingent payoffs into values at date 0

- **Conditional state price density:**
  \[
  \xi_{t,v} \equiv \exp \left( - \int_t^v (r_s + \frac{1}{2} \theta'_s \theta_s) \, ds - \int_t^v \theta'_s \, dW_s \right) = \frac{\xi_v}{\xi_t}
  \]

\[ \pi_t^* = \pi_t^m + \pi_t^r + \pi_t^\theta \]

where

**MV:**
\[ \pi_t^m = \mathbb{E}_t [\xi_{t,T} \Gamma_T^*] (\sigma_t')^{-1} \theta_t \]

**IRH:**
\[ \pi_t^r = - (\sigma_t')^{-1} \mathbb{E}_t \left[ \xi_{t,T} (X_T^* - \Gamma_T^*) \int_t^T D_t r_s ds \right]' \]

**MPRH:**
\[ \pi_t^\theta = - (\sigma_t')^{-1} \mathbb{E}_t \left[ \xi_{t,T} (X_T^* - \Gamma_T^*) \int_t^T (dW_s + \theta_s ds)' D_t \theta_s \right]' \]

- Optimal terminal wealth \( X_T^* = I(y^* \xi_T) \)
- Constant \( y^* \) solves \( x = E [\xi_T I(y^* \xi_T)] \) (static budget constraint)
- \( \Gamma(X) \equiv -U'(X)/U''(X) \): measure of absolute risk tolerance
- \( \Gamma_T^* \equiv \Gamma(X_T^*) \): risk tolerance evaluated at optimal terminal wealth
- \( D_t \) is Malliavin derivative
Structure of Hedges:

IRH: $\pi_t^r = - (\sigma_t^t)^{-1} \mathcal{E}_t \left[ \xi_{t,T} (X_T^* - \Gamma_T^*) \int_t^T D_t r_s ds \right]'$

- Driven by sensitivities of future IR and MPR to current innovations in $W_t$. Sensitivities measured by Malliavin derivatives $D_t r_s$ and $D_t \theta_s$.

- Sensitivities are adjusted by factor $\xi_{t,T} (X_T^* - \Gamma_T^*)$: depends on preferences, terminal wealth and conditional state prices.

- Optimal terminal wealth: $I(y^* \xi_T)$

- Date $t$ cost: $\xi_{t,T} I(y^* \xi_T) = \xi_{t,T} I(y^* \xi_t \xi_{t,T})$

- Sensitivity to change in conditional SPD $\xi_{t,T}$:

$$\frac{\partial (\xi_{t,T} I(y^* \xi_t \xi_{t,T}))}{\partial \xi_{t,T}} = I(y^* \xi_t \xi_{t,T}) + y^* \xi_t \xi_{t,T} I'(y^* \xi_t \xi_{t,T}) = X_T^* - \Gamma_T^*$$

- Sensitivity of conditional SPD to fluctuations in IR and MPR:

$$-\xi_{t,T} \int_t^T D_t r_s ds \quad \text{and} \quad -\xi_{t,T} \int_t^T (dW_s + \theta_s ds)' D_t \theta_s.$$
Constant Relative Risk Aversion (CRRA)

\[ \frac{\pi_t^m}{X_t^*} = \frac{1}{R} (\sigma_t')^{-1} \theta_t \]

\[ \frac{\pi_t^r}{X_t^*} = -\rho (\sigma_t')^{-1} E_t \left[ \frac{\xi_T^\rho}{E_t[\xi_T^\rho]} \int_t^T D_t r_s ds \right]' \]

\[ \frac{\pi_t^\theta}{X_t^*} = -\rho (\sigma_t')^{-1} E_t \left[ \frac{\xi_T^\rho}{E_t[\xi_T^\rho]} \int_t^T (dW_s + \theta_s ds)'D_t \theta_s \right]' \]

- \( \rho = 1 - 1/R \)
- \( y^* = \left( E \left[ \xi_T^\rho \right] / x \right)^R \)
- \( X_t^* = E_t \left[ \xi_t,T (y^* \xi_T)^{-1/R} \right] \)

- Hedging terms are weighted averages of the sensitivities of future interest rates and market prices of risk to the current Brownian innovations.
4 A New Decomposition of the Optimal Portfolio

4.1 Bond Pricing and Forward Measures

- Pure Discount Bond Price: $B^T_t = E_t [\xi_{t,T}]$

- Forward $T$-Measure: (Geman (1989) and Jamshidian (1989))
  - Random variable:
    
    $Z_{t,T} \equiv \frac{\xi_{t,T}}{E_t [\xi_{t,T}]} = \frac{\xi_{t,T}}{B^T_t}$
  - Properties: $Z_{t,T} > 0$ and $E_t [Z_{t,T}] = 1$. Use $Z_{t,T}$ as density
  - Probability measure: $dQ^T_t = Z_{t,T} dP$
    
    $\rightarrow$ Equivalent to $P$
Change of Numéraire: unit of account is $T$-maturity bond

- Under $Q^T_t$ price $V(t)$ of a contingent claim with payoff $Y_T$ is

$$V(t) = E_t [\xi_{t,T} Y_T] = E_t [\xi_{t,T}] E_t \left[ \frac{\xi_{t,T}}{E_t[\xi_{t,T}]} Y_T \right] = B^T_t E^T_t [Y_T]$$

- $E^T_t [\cdot] \equiv E_t [Z_{t,T} \cdot]$ is expectation under $Q^T_t$

- Martingale property: $V(t) / B^T_t = E^T_t [Y_T] = E_t [Z_{t,T} Y_T]$.

- Density $Z_{t,T}$ is stochastic discount factor: converts future payoffs into current values measured in bond unit of account.
Characterization (Theorem 2): The forward $T$-density is given by

$$Z_{t,T} \equiv \exp \left( \int_t^T \sigma^Z (s, T)' dW_s - \frac{1}{2} \int_t^T \sigma^Z (s, T)' \sigma^Z (s, T) ds \right)$$

- volatility at $s \in [t, T]$: $\sigma^Z (s, T) \equiv \sigma^B (s, T) - \theta_s$
- bond return volatility: $\sigma^B (s, T)' \equiv D_s \log B^{T}_s$

Contribution(s):

- Identify volatility of forward measure
- Application of Exponential Clark-Haussmann-Ocone formula
- Market price of risk in the numéraire
4.2 Portfolio allocation and long term bonds

- An Alternative Portfolio Decomposition Formula:

\[ \pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z \]

- Mean variance demand:

\[ \pi_t^m = E_t^T [\Gamma_T^*] B_t^T (\sigma_t')^{-1} \theta_t \]

- Hedge motivated by fluctuations in price of pure discount bond with matching maturity

\[ \pi_t^b = (\sigma_t')^{-1} \sigma_B (t, T) E_t^T [X_T^* - \Gamma_T^*] B_t^T \]

- Hedge motivated by fluctuations in density of forward \( T \)-measure

\[ \pi_t^z = (\sigma_t')^{-1} E_t^T [(X_T^* - \Gamma_T^*) D_t \log (Z_{t,T})] B_t^T. \]
Essence of Formula: change of numéraire

- **SPD representation:** $\xi_{t,T} = B_t^T Z_{t,T}$
- **Optimal terminal wealth:** $X_T^* = I (y^* \xi_t B_t^T Z_{t,T})$
- **Cost of optimal terminal wealth:** $B_t^T Z_{t,T} I (y^* \xi_t B_t^T Z_{t,T})$
- **Hedging portfolio:** $D_t (B_t^T Z_{t,T} I (y^* \xi_t B_t^T Z_{t,T}))$
- **Chain rule of Malliavin calculus:**
  
  $\rightarrow (Z_{t,T} I (y^* \xi_t B_t^T Z_{t,T}) + B_t^T Z_{t,T} I' (y^* \xi_t B_t^T Z_{t,T}) y^* \xi_t Z_{t,T}) D_t B_t^T$

  $\rightarrow (B_t^T I (y^* \xi_t B_t^T Z_{t,T}) + B_t^T Z_{t,T} I' (y^* \xi_t B_t^T Z_{t,T}) y^* \xi_t B_t^T) D_t Z_{t,T}$

  $\rightarrow B_t^T Z_{t,T} I' (y^* \xi_t B_t^T Z_{t,T}) B_t^T Z_{t,T} D_t (y^* \xi_t)$
Long Term Bond Hedge:

- Immunizes against instantaneous fluctuations in return of long term bond with matching maturity date
- Corresponds to portfolio that maximizes the correlation with long term bond return
- This portfolio is a synthetic asset or maturity matching bond itself, if exists

Forward Density Hedge:

- Immunizes against fluctuations in forward density \( Z_{t,T} \) (instantaneous and delayed)
- Source of fluctuations are bond return volatilities and MPRs:
  \[
  \sigma^Z (s, T) \equiv \sigma^B (s, T) - \theta_s
  \]
- \( D_t \sigma^Z (s, T) = D_t \sigma^B (s, T) - D_t \theta_s. \)
Remarks:

- Generality of decomposition is remarkable:
  - Interest rate’s response to Brownian innovations has disappeared
  - Replaced by bond volatilities and MPRs
  - Surprising because infinite dim. Ito processes:
    - Model for prices is not diffusion
    - Current bond prices are not sufficient statistics for IR evolution

- Formula in spirit of immunization strategies sometimes advocated by practitioners
  - First term is static hedge: hedge against current fluctuations in LT bond price
  - To first approximation optimal portfolio has mean-variance term + static hedge
  - Additional hedge fine tunes allocation: captures fluctuations in future quantities
  - Static hedge is preference independent
Signing the Static Hedge:

- Bond prices negatively related to IR
- IR innovation negatively related to equity innovation
- In one factor (BMP) model $\sigma^B > 0$: boost demand for stocks
4.3 Constant Relative Risk Aversion

Hedging Terms are:

\[
\frac{\pi^b_t}{X^*_t} = \rho (\sigma'_t)^{-1} \sigma^B (t, T) B^T_t
\]

\[
\frac{\pi^z_t}{X^*_t} = \rho (\sigma'_t)^{-1} E_t^T \left[ \frac{Z^{\rho-1}_{t,T}}{E_t^T[Z^{\rho-1}_{t,T}]} D_t \log (Z_{t,T}) \right]' B^T_t
\]

Highlights knife-edge property of log utility (Breeden (1979))

- Logarithmic investor displays myopia (hedging demands vanish)
- More (less) risk averse investors will hold (short) portfolio synthesizing long term bond
- More (less) risk averse investors will hold (short) portfolio that hedges forward density
  → portfolio is individual-specific: depends on risk aversion of utility function
Literature: special cases of this result analyzed by


- Lioui and Poncet (2001) and Lioui (2005):
  - Diffusion models with power utility.
  - Lioui and Poncet (2001): last hedging component in terms of unknown volatility function (PDE).
  - Lioui (2005): affine model with mean-reverting IR and MPR processes. Forward density hedge is proportional to vector of volatilities of MPR with proportionality factor linear in MPRs.
Illustration: optimal stock-bond mix for CRRA investor

- Model:
  - $T$-maturity bond is traded
  - Two assets: Stock and investment horizon matching bond

$$
\sigma_t = \begin{bmatrix}
\sigma_{1t}^{stock} & \sigma_{2t}^{stock} \\
\sigma_{1t}^B & \sigma_{2t}^B
\end{bmatrix}
$$

- Optimal portfolio weight: static hedging component

$$
\frac{\pi^b_t}{X^*_t} = \rho (\sigma_t^I)^{-1} \sigma_t^B = \rho \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

- If $\pi^z_t \approx 0$ hedging is very simple:
  - no hedging component for stocks
  - hedging component does not depend on investment horizon
  - hedging portfolio only depends on relative risk aversion coefficient
  - no need to estimate: if risk aversion is $R = 4$, then static hedging component for bonds is 0.75.
Illustration: Asset Allocation Puzzle

- Asset Allocation Puzzle (see Canner, Mankiw and Weil (1997)): investment advisors typically recommend an increase in the bonds-to-equities ratio for more conservative investors while mean-variance portfolio theory predicts that a constant ratio is optimal.

- Bonds-to-equities ratio in Gaussian terms structure models:
  \[
  e(t, T) = \sigma_t S^t \frac{\theta_{2t} + \sigma_{Bt}^T (Q(t, T) - 1)}{\sigma_{Bt}^T \theta_{1t} - \sigma_{Bt}^T \theta_{2t}}
  \]
  \[
  Q(t, T) \equiv \frac{E_t^T [X_T^*]}{E_t^T [\Gamma_T^*]}
  \]

- For HARA utility \( u(x) = (x - A)^{1-R}/1 - R, \)
  \[
  * \quad Q(t, T; R) \equiv R \left( 1 + \left( \frac{B_0^T h(0, t; R)}{x/A - B_0^T} \right) \left( \frac{B_0^T / B_0^T}{B_0^T / B_0^T} \right)^{-1/R} \right) = R \left( 1 + \frac{AB_t^T / X_t^*}{X_t^* - AB_t^T} (B_0^T / B_0^T)^{1/R} \right)
  \]
  * with
    \[
    \cdot \quad h(0, t; R) \equiv \exp \left( \frac{(\rho/R)}{1} \left( \frac{1}{2} \| \theta_s + \sigma^B(s, T) \|^2 - \| \sigma^B(s, T) \|^2 \right) ds \right).
  \]

- Bonds-to-equities ratio risk tolerance, at a given time \( t \), if and only if \( Q(t, T) \) is a monotone function of risk tolerance.
• In the presence of wealth effect the bonds-to-equities ratio is not necessarily monotone in risk aversion

– Vasicek interest rate model: $r_0 = \bar{r} = 0.06$, $\kappa_r = 0.05$, $\sigma_{r1} = -0.02$, $\sigma_{r2} = -0.015$ and market prices of risk are constants $\theta_s = 0.3$ and $\theta_b = 0.15$. The interest rate at $t = 5$ is $r_t = 0.02$.

Other parameter values are $A = 200,000$, $x = 100,000$ and $T = 10$. 
5 Intermediate Consumption

5.1 The Investor’s Preferences

- **Consumption-portfolio Problem:**

\[
\max_{\pi, c \in A} \mathbb{E} \left[ \int_0^T u(c_t, t) \, dt + U(X_T) \right]
\]

- **Utility function:** \( u(\cdot, \cdot) : \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R} \) and bequest function: \( U : \mathbb{R}_+ \rightarrow \mathbb{R} \) satisfy standard assumptions

- Maximization over set of admissible portfolio policies \( \pi, c \in A \)

- Inverse marginal utility function \( J(y, t) \) exists: \( u' \left( J(y, t), t \right) = y \) for all \( t \in [0, T] \)

- Inverse marginal bequest function \( I(y) \) exists: \( U' \left( I(y) \right) = y \)
5.2 Portfolio Representation and Coupon-paying Bonds

- **Decomposition:**
  \[ \pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z \]

- **Mean variance demand:**
  \[ \pi_t^m = \left( \int_t^T E_t^v \left[ \Gamma_v^* \right] B_t^v dv + E_t^T \left[ \Gamma_T^* \right] B_t^T \right) (\sigma_t')^{-1} \theta_t \]

- **Hedge** motivated by fluctuations in price of **coupon-paying bond** with matching maturity:
  \[ \pi_t^b = (\sigma_t')^{-1} \int_t^T \sigma^B(t, v) B_t^v E_t^v \left[ c_v^* - \Gamma_v^* \right] dv \]
  \[ + (\sigma_t')^{-1} \sigma^B(t, T) B_t^T E_t^T \left[ X_T^* - \Gamma_T^* \right] \]

- **Hedge** motivated by fluctuations in **density of forward** **$T$-measure:**
  \[ \pi_t^z = (\sigma_t')^{-1} \left( \int_t^T E_t^v \left[ (c_v^* - \Gamma_v^*) D_t \log Z_{t,v} \right] B_t^v dv \right)' \]
  \[ + (\sigma_t')^{-1} \left( E_t^T \left[ (X_T^* - \Gamma_T^*) D_t \log Z_{t,T} \right] B_t^T \right)' \]
Static Hedge $\pi^b_t$: hedge against fluctuations in value of coupon-paying bond

- **Coupon payments** $C(v) \equiv E_t^v [c^*_v - \Gamma^*_v]$ at intermediate dates $v \in [0, T)$
- **Bullet payment** $F \equiv E_t^T [X^*_T - \Gamma^*_T]$ at terminal date $T$
- Coupon payments and face value are
time-varying
tailored to individual’s consumption profile and risk tolerance

- **Bond value**

\[
B(t, T; C, F) \equiv \int_t^T B_t^v C(v) \, dv + B_t^T F.
\]

- **Instantaneous volatility**

\[
\sigma(B(t, T; C, F)) B(t, T; C, F) = \int_t^T \sigma^B(t, v) B_t^v C(v) \, dv + \sigma^B(t, T) B_t^T F
\]

- **Hedge**: $(\sigma'_t)^{-1} \sigma(B(t, T; C, F)) B(t, T; C, F)$
Forward Density Hedge $\pi^\varepsilon_t$:

- **Motivation:** desire to hedge *fluctuations in forward densities* $Z_{t,v}$
- **Static hedge** already neutralizes impact of term structure fluctuations on PV of future consumption
- **Given** $\xi_{t,v} = B_{t}^{v}Z_{t,v}$ it remains to hedge *fluctuations in risk-adjusted discount factors* $Z_{t,v}, v \in [t, T]$.

Optimal Portfolio Composition:

- **To first approximation** optimal portfolio has mean-variance term + long term coupon bond hedge
- **Under what conditions** is this approximation exact (i.e. last term vanishes)?
- **If last term** does not vanish what is its size?
5.3 Constant Relative Risk Aversion

- Relative risk aversion parameters $R_u, R_U$ for utility and bequest functions. Portfolio:
  
  - Mean-variance term
    
    $$\pi^m_t = (\sigma'_t)^{-1} \left( \int_t^T \frac{1}{R_u} E^v_t [c^*_v] B^v_t dv + \frac{1}{R_U} E^T_t [X^*_T] B^T_t \right) \theta_t$$

  - Hedge motivated by fluctuations in price of coupon-paying bond with matching maturity
    
    $$\pi^b_t = (\sigma'_t)^{-1} \left( \rho_u \int_t^T \sigma^B (t,v) B^v_t E^v_t [c^*_v] dv + \rho_U \sigma^B (t,T) B^T_t E^T_t [X^*_T] \right)$$

  - Hedge motivated by fluctuations in densities of forward measures
    
    $$\pi^z_t = \rho_u (\sigma'_t)^{-1} \int_t^T E^v_t [c^*_v D_t \log Z_{t,v}]' B^v_t dv + \rho_U (\sigma'_t)^{-1} E^T_t [X^*_T D_t \log Z_{t,T}]' B^T_t$$
Static Hedge has two parts:

- Pure coupon bond (annuity) with coupon given by optimal consumption
- Bullet payment given by optimal terminal wealth
- Two parts are weighted by risk aversion factors $\rho_u$ and $\rho_U$
- Knife edge property traditionally associated with power utility function
- Possibility of positive annuity hedge ($R_u > 1$) combined with negative bequest hedge ($R_U < 1$).
6 Applications

6.1 Preferred Habitats and Portfolio Choice

 Preferred Habitat Theory Modigliani and Sutch (1966):

- Individuals exhibit preference for securities with maturities matching their investment horizon
- Investor who cares about terminal wealth should invest in bonds with matching maturity
- Existence of group of investors with common investment horizon might lead to increase in demand for bonds in this maturity range
- Implies increase in bond prices and decrease in yields. Explains hump-shaped yield curves with decreasing profile at long maturities.
Formula shows that optimal behavior naturally induces a demand for certain types of bonds in specific maturity ranges

\[
\pi^*_t = w^m_t (X^*_t - B(t, T; C, F)) + w^b_t B(t, T; C, F) + \pi^*_z
\]

where

\[
\pi^*_z = \arg \max_{\pi} \{ \pi' \sigma_t \hat{\sigma}(t, T) : \pi' \sigma_t \sigma'_t \pi' = k \}
\]

\[
\hat{\sigma}'_{t,T} \equiv \int_t^T E_t^v [ (c^*_v - \Gamma^*_v) D_t \log Z_{t,v} ] B_v^v dv
+ E_t^T [ (X^*_T - \Gamma^*_T) D_t \log Z_{t,T} ] B_t^T
\]
• Any individual has preferred bond habitat:
  → Optimal portfolio includes long term bond with maturity date matching the investor’s horizon
  → Preferred instrument is coupon-paying bond with payments tailored to consumption profile of investor

• Complemented by mean-variance efficient portfolio to constitute the static component of allocation

• Under general market conditions the static policy is fine-tuned by dynamic hedge
  → When bond return volatilities and market prices of risk are deterministic, dynamic hedge vanishes
Equilibrium Implications

- Existence of natural preferred habitats in certain segments of fixed income market
- Existence of equilibrium effects on prices and premia in these habitats depends on characteristics of investors’ population
- With sufficient homogeneity
  - Strong demand for structured fixed income products might emerge
  - Prompt financial institutions to offer tailored products appealing to those segments
  - Yields to maturity would then naturally reflect this habitat-motivated demand
Motivation for preferred habitat here is different from Riedel (2001)

- In his model habitat preferences are driven by structure of subjective discount rates placing emphasis on specific future dates
- In our setting preference for long term bonds emerges from the structure of the hedging terms
- Optimal hedging combines static hedge (long term bond) with dynamic hedge motivated by fluctuations in forward measure volatilities
6.2 Universal Fund Separation

- $Y$: vector of $N < d$ state variables with evolution described by the functional stochastic differential equation

$$dY_t = \mu(Y_{\cdot})_t dt + \sigma(Y_{\cdot})_t dW_t$$

- Suppose that
  - $B^v_t = B(t, v, Y_{\cdot})$
  - $\sigma^Z(t, v) = \sigma^Z(t, v, Y_{\cdot})$

are Fréchet differentiable functionals of $Y_{\cdot}$.

- Universal $N + 1$-fund separation holds: portfolio demands can be synthesized by investing in $N + 2$ (preference free) mutual funds:
  1. riskless asset
  2. the mean-variance efficient portfolio
  3. $N$ portfolios $(\sigma'_t)^{-1} \sigma^Y_t (Y_{\cdot})'$ to synthesize the static bond hedge and the forward density hedge.
6.3 Extreme Behavior

Assume risk tolerances go to zero:

- **Intermediate utility and bequest functions:**
  \[(\Gamma_u(z,v), \Gamma_U(z)) \to (0,0) \text{ for all } z \in [0, +\infty) \text{ and all } v \in [0, T]\]

- **Relative behaviors:** for some constant \( k \in [0, +\infty) \):
  \[
  \frac{\Gamma_u(z_1,v)}{\Gamma_u(z_2)} \to k \text{ for all } z_1, z_2 \in [0, \infty) \text{ and all } v \in [0, T]
  \]
  \[
  \frac{\Gamma_u(z_1,v_1)}{\Gamma_u(z_2,v_2)} \to 1 \text{ for all } z_1, z_2 \in [0, \infty) \text{ and all } v_1, v_2 \in [0, T]
  \]
Limit Allocations: coupon-paying bond with constant coupon \( C \) and face value \( F \) given by

\[
C = \frac{x}{\int_0^T B_0^v dv + B_0^T / k} \quad \text{and} \quad F = \frac{x}{\int_0^T B_0^v dv k + B_0^T}.
\]

- If \( k = 0 \) exclusive preference for pure discount bond,
  \((C, F) = (0, x/B_0^T)\)
- If \( k \to \infty \) preference is for a pure coupon bond,
  \((C, F) = \left( x/\int_0^T B_0^v dv, 0 \right)\)

Limit Behavior:

- Governed by relation between utility functions at different dates
- As risk tolerances vanish, preference for certainty: coupon-paying bond with bullet payment
- Least extreme of the extreme behaviors drives the habitat:
  - Given a preference for riskless instruments: individuals puts more weight on maturities where risk tolerance is greater
  - Exhibits a time preference in the limit.
Illustration: CARA preferences $\Gamma_u$ and $\Gamma_U$ constant, $k \equiv \Gamma_u / \Gamma_U$.

- Slope of indifference curves:

$$- \frac{dX}{dc} = \frac{1}{k} \left( e^{X-c/k} \right) \frac{1}{\Gamma_U}$$
\[ k \rightarrow 0 \]
• $k \rightarrow \infty$
Special case examined by Wachter (2002)

- Arbitrary utility functions over terminal wealth and markets with general coefficients
- Documents emergence of preferred habitat when relative risk aversion goes to infinity
  → Pure discount bond with unit face value and matching maturity
- Our analysis shows that preferred habitat for an extreme consumer may take different forms depending on nature of behavior
  → Pure discount bonds, pure annuities or coupon-paying bonds with bullet payments at maturity can emerge in limit.
- As \( (\Gamma_u(z, v), \Gamma_U(z)) \to (0, 0) \), the limit portfolios
  
  \[ \pi^m_t = \pi^z_t = 0 \]
  
  \[ \pi^b_t = (\sigma'_t) \int_0^T \sigma^B(t, v) B^v_t dv C + \sigma^B(t, T) B^T_t F \]

- have scaled asymptotic errors:

  \[ \epsilon^\alpha_t(\nu) = (\Gamma_\nu(\cdot))^{-1} (\pi^\alpha_t - \pi^\alpha_t) \quad \text{with} \quad \alpha \in \{m, b, z\} \quad \text{and} \quad \nu \in \{u, U\}, \]

\[
\begin{align*}
[\epsilon_t^m(U), \epsilon_t^m(u)] & \to (\sigma'_t)^{-1} \theta_t \left[ \int_t^T B^v_t dv B^T_t \right] K \\
[\epsilon_t^b(U), \epsilon_t^b(u)] & \to - (\sigma'_t)^{-1} \left[ \int_t^T \sigma^B(t, v) B^v_t dv \sigma^B(t, T) B^T_t \right] K \\
[\epsilon_t^z(U), \epsilon_t^z(u)] & \to - (\sigma'_t)^{-1} \left[ \int_t^T N_{t,v} B^v_t dv N_{t,T} B^T_t \right] K 
\end{align*}
\]

- where

  \[ N_{t,\tau} \text{ is given by} \]

\[
N_{t,\tau} \equiv E_t^\tau \left[ \left( \int_t^\tau \sigma^Z(r, \tau)' dW_r - \frac{1}{2} \int_t^\tau \| \sigma^Z(r, \tau) \|^2 dr \right) (D_t \log Z_{t,\tau})' \right]
\]

  \[ K \text{ is given by} \]

\[
K \equiv \begin{bmatrix} k & 1 \\ 1 & \frac{1}{k} \end{bmatrix}
\]
Integration of term structure models and asset allocation models:

- Forward rate representation of bonds

\[ B_t^v = \exp \left( - \int_t^v f_t^s ds \right) \]

- Continuously compounded forward rate:

\[ f_t^s = - \frac{\partial}{\partial v} \log (B_t^v) \]

- Bond price volatility:

\[ \sigma^B(t, v)' = D_t \log B_t^v = - \int_t^v D_t f_t^s ds = - \int_t^v \sigma^f(t, s) ds \]

- Volatility of forward rate: \( \sigma^f(t, s) \)

- Forward rate dynamics:

- No arbitrage condition (HJM (1992)):

\[ df_t^v = \sigma^f(t, v) \left( dW_t + (\theta_t - \sigma^B(t, v)) dt \right), \quad f_0^v \text{ given} \]

- Dynamics completely determined by forward rate volatility function and initial forward rate curve
Optimal Portfolio: previous formula with

\[
\mathcal{D}_t \log Z_{t,v} = \int_t^v \left( dW_s + \left( \theta_s + \int_s^v \sigma^f(s,u) \, du \right) \, ds \right)' \left( \mathcal{D}_t \theta_s + \int_s^v \mathcal{D}_t \sigma^f(s,u) \, du \right)
\]

- Forward density hedge in terms of forward rate volatilities
- Useful for financial institution using a specific HJM model to price/hedge fixed income instruments and their derivatives
- Implied forward rates inferred from term structure model and observed prices
  - \( \rightarrow \) estimate volatility function \( \sigma^f(s,u) \)
  - \( \rightarrow \) feed into asset allocation formula
- Simple integration of fixed income management and asset allocation.
**Forward Density Hedge:**

- Immunization demand due to fluctuations in future market prices of risk and forward rate volatilities
- Vanishes if deterministic forward rate volatilities $\sigma^f(s,u)$ and market prices of risk $\theta_s$
- Pure expectation hypothesis holds under forward measure:
  $$f(t,v) = E^v_t[r_v]$$
  → Standard version of PEH ($f(t,v) = E_t[r_v]$) fails when $Z_{t,v} \neq 1$
  → Density process $Z_{t,v}$ measures deviation from PEH
  → Malliavin derivative $D_t \log Z_{t,v}$ captures sensitivity of deviation with respect to shocks
  → Dynamic hedge = hedge against deviations from PEH
  → If $Z_{t,v} = 1$ PEH holds under the original beliefs and hedging becomes irrelevant
  → If $\sigma^Z$ deterministic, deviations from PEH are non-predictable and do not need to be hedged
- Extensively employed in practice
- Forward rate volatilities $\sigma^f$ are insensitive to shocks. If MPR also deterministic no need to hedge
- Bajeux-Besnainou, Jordan and Portait (2001) also falls in this category (one factor Vasicek)
Numerical Results: Forward measure hedges in one factor CIR model

- CIR interest rates:

\[
dr_t = \kappa_r (\bar{r} - r_t)\,dt + \sigma_r \sqrt{r} \,dW_t; \quad r_0 = r
\]

\[\rightarrow\] Parameter values (Durham (JFE, 2003)):

- \(\kappa_r = 0.002\)
- \(\bar{r} = 0.0497\)
- \(\sigma_r = -0.0062\)
- \(r = 0.06\)

- Market price of risk:

\[
\theta_t = \gamma_r \sqrt{r_t}
\]

\[\rightarrow\] Parameter values:

- \(\gamma_r = 0.3/\sqrt{\bar{r}} \) such that \(\bar{\theta} = \gamma_r \sqrt{\bar{r}} = 0.3\)

- CRRA preferences for terminal wealth
• Mean-variance demand: \[ \frac{\pi_t^{mv}}{X_t^*} = \frac{1}{R} (\sigma'_t)^{-1} \theta_t \]
• Static term structure hedge: 
\[ \frac{\pi_t^b}{X_t^*} = \rho(\sigma_t')^{-1}\sigma^B(t, T) \]
- Dynamic forward measure hedge:

$$\pi_t^{\hat{z}} / X_t^* = \rho (\sigma_t')^{-1} \mathbb{E}_t^T \left[ \frac{Z_{t,T}^{\rho^{-1}}}{\mathbb{E}_t^T[Z_{t,T}^{\rho^{-1}}]} (D_t \log Z_{t,T})' \right]$$
• Total portfolio weight: \[ \frac{\pi_t}{X_t^*} = \frac{\pi_t^{mv}}{X_t^*} + \frac{\pi_t^b}{X_t^*} + \frac{\pi_t^z}{X_t^*} \]
• Changing initial interest rate: Relative risk aversion fixed at $R = 4$

→ Mean-variance demand: 
\[ \pi_t^{mv} / X_t^* = \frac{1}{R} (\sigma'_t)^{-1} \theta_t \]
→ Static term structure hedge: $\pi_t^b / X_t^* = \rho(\sigma'_t)^{-1} \sigma^B(t, T)$
Dynamic forward measure hedge:

\[ \frac{\pi_t^z}{X_t^z} = \rho \left( \sigma_t' \right)^{-1} \mathbb{E}_t^T \left[ \frac{Z_{t,T}^{\rho-1}}{\mathbb{E}_t^T[Z_{t,T}^{\rho-1}]} \left( \mathcal{D}_t \log Z_{t,T} \right)' \right] \]
- Total portfolio weight: \( \pi_t / X_t^* = \pi_t^{mv} / X_t^* + \pi_t^b / X_t^* + \pi_t^z / X_t^* \)
Changing initial interest rate: Investment horizon fixed at $T = 15$

Mean-variance demand: $\frac{\pi_t^{mv}}{X_t^{ast}} = \frac{1}{R} (\sigma'_t)^{-1} \theta_t$
→ Static term structure hedge: \[ \frac{\pi_t^b}{X_t^*} = \rho(\sigma'_t)^{-1}\sigma^B(t, T) \]
Dynamic forward measure hedge:

\[
\pi^z_t / X^*_t = \rho (\sigma'_t)^{-1} \mathbb{E}_t^T \left[ \frac{Z_{t,T}^{\rho -1}}{\mathbb{E}_t^T \left[ Z_{t,T}^{\rho -1} \right]} (D_t \log Z_{t,T})' \right]
\]
• Total portfolio weight: 

\[ \pi_t/X_t^* = \pi_t^{mv}/X_t^* + \pi_t^b/X_t^* + \pi_t^z/X_t^* \]
• Approximate forward density portfolio weight: 

\[ \pi_t^{fa} / X_t^* = -\sigma_t^{-1} \frac{1}{R_t} E_t^T [N_t, T] \]
• Approximate forward density portfolio weight: \( \pi_t^f / X_t^* = -\sigma_t^{-1} \frac{1}{R_t} E_t^T [N_t, T] \)
7 Conclusion

► Contributions:

• Asset allocation formula based on change of numéraire
• Highlights role of consumption-specific coupon bonds as instruments to hedge fluctuations in opportunity set
• Formula has multiple applications: preferred habitat, extreme behavior, international asset allocation, demand for I-bonds
• Exponential Clark-Haussmann-Ocone formula
• Malliavin derivatives of functional SDEs
• Solution of linear BVIE

► Integration of portfolio management and term structure models

• Asset allocation in HJM framework
• Other applications

► Universal $N + 2$ fund separation result