Dynamic Asset Allocation: a Portfolio Decomposition Formula and Applications

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1 Introduction

- Dynamic consumption-portfolio choice:
  - Merton (1971): optimal portfolio includes intertemporal hedging terms in addition to mean-variance component (diffusion)
  - Breeden (1979): hedging performed by holding funds giving best protection agst fluctuations in state variable (diffusion)
  - Ocone and Karatzas (1991): representation of hedging terms using Malliavin derivatives (Ito, complete markets)

    → Interest rate hedge

    → Market price of risk hedge
  - Detemple, Garcia and Rindisbacher (DGR JF, 2003): practical implementation of model (diffusion, complete markets)

    → Based on Monte Carlo Simulation

    → Flexible method: arbitrary # assets and state variables, non-linear dynamics, arbitrary utility functions

    → Extends to incomplete/frictional markets (DR MF, 2005)

- Contribution:
  - **New decomposition of optimal portfolio (hedging terms):**

  → Formula rests on change of numéraire: use pure discount bonds as units of account

  → Passage to a new probability measure: forward measure (Geman (1989) and Jamshidian (1989))

  → General context: Ito price processes, general utilities

- **New economic insights about structure of hedges:**
  - → Utility from terminal wealth: hedge
    - · fluctuations in instantaneaous price of long term bond with maturity date matching investment horizon
    - · fluctuations in future bond return volatilities and future market prices of risk (forward density)
    - · first hedge has a static flavor (static hedge)
  - → Utility from terminal wealth and intermediate consumption
    - · static hedge is a coupon-paying bond, with variable coupon payments tailored to consumption needs
  - → Risk aversion properties:
    - · if risk aversion approaches one both hedges vanish: myopia
    - · if risk aversion becomes large mean-variance term and second hedge vanish: holds just long term bonds
    - · if risk aversion vanishes all terms are of first order in risk tolerance.
  - → Non-Markovian $N + 2$ fund separation theorem.

- **Technical contribution:**

  → Exponential version of Clark-Haussmann-Ocone formula

  → Identifies volatilities of exponential martingale in terms of Malliavin derivatives

  → Malliavin derivatives of functional SDEs

  → Explicit solution of a Backward Volterra Integral Equation (BVIE) involving Malliavin derivatives.

- Applications:
  - Preferred habitat
  - Extreme risk aversion behavior
  - International asset allocation
  - Preferences for I-bonds
  - Integration of risk management and asset allocation
- Road map:
  - Model with utility from terminal wealth
  - The Ocone-Karatzas formula
  - New representation
  - Intermediate consumption
  - Applications
  - Conclusions

2 The Model

- Standard Continuous Time Model:
  - Complete markets and Ito price processes
  - Brownian motion W, d-dimensional
  - Flow of information $\mathcal{F}_t = \sigma(W_s : s \in [0, t])$
  - Finite time period $[0, T]$.
  - Possibly non-Markovian dynamics

► Assets: Price Evolution

- **Risky assets (dividend-paying assets):**

$$\boxed{\frac{dS_t^i}{S_t^i} = \left(r_t - \delta_t^i\right) dt + \sigma_t^i \left(\theta_t dt + dW_t\right), \quad S_0^i \text{ given}}$$

  * σ_t^i: volatility coefficients of return process ($1 \times d$ vector)
  * r_t: instantaneous rate of interest
  * δ_t^i: dividend yield
  * θ_t : market prices of risk associated with W ($d \times 1$ vector)
  * $(r, \delta, \sigma, \theta)$: progressively measurable processes; standard integrability conditions

- **Riskless asset:**
  * pays interest at rate r

- Investment and Wealth:
  - **Portfolio policy π:** d-dimensional, progressively measurable; integrability conditions

    → amounts invested in assets: π

    → amount in money market: $X - \pi' \mathbf{1}$
  - **Wealth process:**

$$\boxed{dX_t = r_t X_t dt + \pi_t' \sigma_t \left(\theta_t dt + dW_t \right), \text{ subject to } X_0 = x.}$$

  - **Admissibility:** π is admissible ($\pi \in \mathcal{A}$) if and only if wealth is non-negative: $X \geq 0$.

► Asset Allocation Problem:

- **Investor maximizes expected utility of terminal wealth:**

$$\boxed{\max_{\pi\in\mathcal{A}} \mathbf{E}\left[U(X_T)\right]}$$

- **Utility function:** $U:\mathbb{R}_+ \to \mathbb{R}$

$\rightarrow$ Strictly increasing, strictly concave and differentiable

$\rightarrow$ Inada conditions: $\lim_{X\to\infty} U'(X) = 0$ and $\lim_{X\to 0} U'(X) = \infty$

- Includes **CRRA** $U(x) = \frac{1}{1-R} X^{1-R}$ where $R > 0$.

- **Property:**

$\rightarrow$ Strictly decreasing marginal utility in $(0,\infty)$

$\rightarrow$ Inverse marginal utility $I\left(y\right)$ exists and satisfies $U'\left(I\left(y\right)\right) = y$

$\rightarrow$ Derivative: $I'(y) = 1/U''(I(y))$

3 The Optimal Portfolio

- Complete Markets:

  - Market price of risk: $\theta_t = (\theta_{1t}, ..., \theta_{dt})'$

  - **State price density:**

$$\boxed{\xi_v \equiv \exp\left(-\int_0^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_0^v \theta_s' dW_s\right)}$$

  $\rightarrow$ converts state-contingent payoffs into values at date 0

  - **Conditional state price density:**

$$\boxed{\xi_{t,v} \equiv \exp\left(-\int_t^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_t^v \theta_s' dW_s\right) = \xi_v / \xi_t}$$

- Optimal Portfolio: Ocone and Karatzas (1991), Detemple, Garcia and Rindisbacher (2003)

$$\pi_t^* = \pi_t^m + \pi_t^r + \pi_t^\theta$$

where

MV: $\pi_t^m = \mathbf{E}_t\left[\xi_{t,T}\Gamma_T^*\right]\left(\sigma_t'\right)^{-1}\theta_t$

IRH: $\pi_t^r = -\left(\sigma_t'\right)^{-1}\mathbf{E}_t\left[\xi_{t,T}\left(X_T^* - \Gamma_T^*\right)\int_t^T \mathcal{D}_t r_s ds\right]'$

MPRH: $\pi_t^\theta = -\left(\sigma_t'\right)^{-1}\mathbf{E}_t\left[\xi_{t,T}\left(X_T^* - \Gamma_T^*\right)\int_t^T \left(dW_s + \theta_s ds\right)' \mathcal{D}_t\theta_s\right]'$

- Optimal terminal wealth $X_T^* = I(y^*\xi_T)$
- Constant y^* solves $x = E\left[\xi_T I(y^*\xi_T)\right]$ (static budget constraint)
- $\Gamma(X) \equiv -U'(X)/U''(X)$: measure of absolute risk tolerance
- $\Gamma_T^* \equiv \Gamma\left(X_T^*\right)$: risk tolerance evaluated at optimal terminal wealth
- $\mathcal{D}_t$ is Malliavin derivative

- Structure of Hedges:

IRH: $$\pi_t^r = -\left(\sigma_t'\right)^{-1} \mathbf{E}_t \left[\xi_{t,T} \left(X_T^* - \Gamma_T^*\right) \int_t^T \mathcal{D}_t r_s ds\right]'$$

- Driven by sensitivities of future IR and MPR to current innovations in W_t. Sensitivities measured by Malliavin derivatives $\mathcal{D}_t r_s$ and $\mathcal{D}_t \theta_s$
- Sensitivities are adjusted by factor $\xi_{t,T} \left(X_T^* - \Gamma_T^*\right)$: depends on preferences, terminal wealth and conditional state prices.
- Optimal terminal wealth: $I(y^* \xi_T)$
- Date t cost: $\xi_{t,T} I(y^* \xi_T) = \xi_{t,T} I(y^* \xi_t \xi_{t,T})$
- Sensitivity to change in conditional SPD $\xi_{t,T}$

$$\frac{\partial\left(\xi_{t,T} I(y^* \xi_t \xi_{t,T})\right)}{\partial \xi_{t,T}} = I(y^* \xi_t \xi_{t,T}) + y^* \xi_t \xi_{t,T} I'(y^* \xi_t \xi_{t,T}) = X_T^* - \Gamma_T^*$$

- Sensitivity of conditional SPD to fluctuations in IR and MPR

$$-\xi_{t,T} \int_t^T \mathcal{D}_t r_s ds \quad \text{and} \quad -\xi_{t,T} \int_t^T \left(dW_s + \theta_s ds\right)' \mathcal{D}_t \theta_s.$$

- Constant Relative Risk Aversion (CRRA)

$$\frac{\pi_t^m}{X_t^*} = \frac{1}{R}\left(\sigma_t'\right)^{-1}\theta_t$$

$$\frac{\pi_t^r}{X_t^*} = -\rho\left(\sigma_t'\right)^{-1}\mathbf{E}_t\left[\frac{\xi_T^\rho}{\mathbf{E}_t\left[\xi_T^\rho\right]}\int_t^T \mathcal{D}_t r_s ds\right]'$$

$$\frac{\pi_t^\theta}{X_t^*} = -\rho\left(\sigma_t'\right)^{-1}\mathbf{E}_t\left[\frac{\xi_T^\rho}{\mathbf{E}_t\left[\xi_T^\rho\right]}\int_t^T (dW_s + \theta_s ds)'\mathcal{D}_t\theta_s\right]'$$

- $\rho = 1 - 1/R$
- $y^* = \left(\mathbf{E}\left[\xi_T^\rho\right]/x\right)^R$
- $X_t^* = \mathbf{E}_t\left[\xi_{t,T}(y^*\xi_T)^{-1/R}\right]$
- Hedging terms are weighted averages of the sensitivities of future interest rates and market prices of risk to the current Brownian innovations.

4 A New Decomposition of the Optimal Portfolio

4.1 Bond Pricing and Forward Measures

- Pure Discount Bond Price: $B_t^T = E_t\left[\xi_{t,T}\right]$
- Forward T-Measure: (Geman (1989) and Jamshidian (1989))
  - **Random variable:**

$$\boxed{Z_{t,T} \equiv \frac{\xi_{t,T}}{E_t\left[\xi_{t,T}\right]} = \frac{\xi_{t,T}}{B_t^T}}$$

  - **Properties:** $Z_{t,T} > 0$ and $E_t\left[Z_{t,T}\right] = 1$. Use $Z_{t,T}$ as density
  - **Probability measure:** $dQ_t^T = Z_{t,T} dP$

  $\rightarrow$ Equivalent to P

- Change of Numéraire: unit of account is T-maturity bond
  - Under Q_t^T price $V(t)$ of a contingent claim with payoff Y_T is

$$V(t) = E_t[\xi_{t,T} Y_T] = E_t[\xi_{t,T}] E_t\left[\frac{\xi_{t,T}}{E_t[\xi_{t,T}]} Y_T\right] = B_t^T E_t^T[Y_T]$$

  - $E_t^T[\cdot] \equiv E_t[Z_{t,T}\cdot]$ is expectation under Q_t^T
  - **Martingale property:** $V(t)/B_t^T = E_t^T[Y_T] = E_t[Z_{t,T} Y_T]$.
  - Density $Z_{t,T}$ is **stochastic discount factor:** converts future payoffs into current values measured in bond unit of account.

- **Characterization (Theorem 2):** The forward T-density is given by

$$Z_{t,T} \equiv \exp\left(\int_t^T \sigma^Z(s,T)' \, dW_s - \frac{1}{2} \int_t^T \sigma^Z(s,T)' \, \sigma^Z(s,T) \, ds \right)$$

  - **volatility at** $s \in [t,T]$**:** $\sigma^Z(s,T) \equiv \sigma^B(s,T) - \theta_s$
  - **bond return volatility:** $\sigma^B(s,T)' \equiv \mathcal{D}_s \log B_s^T$

- **Contribution(s):**
  - Identify **volatility of forward measure**
  - Application of **Exponential Clark-Haussmann-Ocone formula**
  - **Market price of risk in the numéraire**

4.2 Portfolio allocation and long term bonds

- An Alternative Portfolio Decomposition Formula:

$$\boxed{\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z}$$

  - **Mean variance demand:**

$$\boxed{\pi_t^m = E_t^T\left[\Gamma_T^*\right] B_t^T \left(\sigma_t'\right)^{-1} \theta_t}$$

  - **Hedge** motivated by fluctuations in price of **pure discount bond with matching maturity**

$$\boxed{\pi_t^b = \left(\sigma_t'\right)^{-1} \sigma^B\left(t,T\right) E_t^T\left[X_T^* - \Gamma_T^*\right] B_t^T}$$

  - **Hedge** motivated by fluctuations in **density of forward** T**-measure**

$$\boxed{\pi_t^z = \left(\sigma_t'\right)^{-1} E_t^T\left[\left(X_T^* - \Gamma_T^*\right) \mathcal{D}_t \log\left(Z_{t,T}\right)\right]' B_t^T.}$$

▶ Essence of Formula: change of numéraire

- **SPD representation:** $\xi_{t,T} = B_t^T Z_{t,T}$
- **Optimal terminal wealth:** $X_T^* = I\left(y^* \xi_t B_t^T Z_{t,T}\right)$
- **Cost of optimal terminal wealth:** $B_t^T Z_{t,T} I\left(y^* \xi_t B_t^T Z_{t,T}\right)$
- **Hedging portfolio:** $\mathcal{D}_t\left(B_t^T Z_{t,T} I\left(y^* \xi_t B_t^T Z_{t,T}\right)\right)$
- **Chain rule of Malliavin calculus:**

$$\boxed{\rightarrow \left(Z_{t,T} I\left(y^* \xi_t B_t^T Z_{t,T}\right) + B_t^T Z_{t,T} I'\left(y^* \xi_t B_t^T Z_{t,T}\right) y^* \xi_t Z_{t,T}\right) \mathcal{D}_t B_t^T}$$

$$\boxed{\rightarrow \left(B_t^T I\left(y^* \xi_t B_t^T Z_{t,T}\right) + B_t^T Z_{t,T} I'\left(y^* \xi_t B_t^T Z_{t,T}\right) y^* \xi_t B_t^T\right) \mathcal{D}_t Z_{t,T}}$$

$$\boxed{\rightarrow B_t^T Z_{t,T} I'\left(y^* \xi_t B_t^T Z_{t,T}\right) B_t^T Z_{t,T} \mathcal{D}_t\left(y^* \xi_t\right)}$$

- **Long Term Bond Hedge:**
  - Immunizes against instantaneous fluctuations in return of long term bond with matching maturity date
  - Corresponds to portfolio that maximizes the correlation with long term bond return
  - This portfolio is a synthetic asset or maturity matching bond itself, if exists
- **Forward Density Hedge:**
  - Immunizes against fluctuations in forward density $Z_{t,T}$ (instantaneous and delayed)
  - Source of fluctuations are bond return volatilities and MPRs: $\sigma^{Z}\left(s,T\right) \equiv \sigma^{B}\left(s,T\right) - \theta_{s}$
  - $\mathcal{D}_{t}\sigma^{Z}\left(s,T\right) = \mathcal{D}_{t}\sigma^{B}\left(s,T\right) - \mathcal{D}_{t}\theta_{s}$.

- Remarks:
  - Generality of decomposition is remarkable:
    - → Interest rate's response to Brownian innovations has disappeared
    - → Replaced by bond volatilities and MPRs
    - → Surprising because infinite dim. Ito processes:
      - · Model for prices is not diffusion
      - · Current bond prices are not sufficient statistics for IR evolution
  - Formula in spirit of immunization strategies sometimes advocated by practitioners
    - → First term is static hedge: hedge against current fluctuations in LT bond price
    - → To first approximation optimal portfolio has mean-variance term + static hedge
    - → Additional hedge fine tunes allocation: captures fluctuations in future quantities
    - → Static hedge is preference independent

- Signing the Static Hedge:
  - Bond prices negatively related to IR
  - IR innovation negatively related to equity innovation
  - In one factor (BMP) model $\sigma^B > 0$: boost demand for stocks

4.3 Constant Relative Risk Aversion

- Hedging Terms are:

$$\frac{\pi_t^b}{X_t^*} = \rho \left(\sigma_t'\right)^{-1} \sigma^B(t,T)\, B_t^T$$

$$\frac{\pi_t^z}{X_t^*} = \rho \left(\sigma_t'\right)^{-1} E_t^T \left[\frac{Z_{t,T}^{\rho-1}}{E_t^T\left[Z_{t,T}^{\rho-1}\right]} \mathcal{D}_t \log\left(Z_{t,T}\right) \right]' B_t^T$$

- Highlights knife-edge property of log utility (Breeden (1979))
  - Logarithmic investor displays myopia (hedging demands vanish)
  - More (less) risk averse investors will hold (short) portfolio synthesizing long term bond
  - More (less) risk averse investors will hold (short) portfolio that hedges forward density

  → portfolio is individual-specific: depends on risk aversion of utility function

- Literature: special cases of this result analyzed by
  - Bajeux-Besnainou, Jordan and Portait (2001): Vasicek short rate and constant market prices of risk. Forward density hedge vanishes
  - Lioui and Poncet (2001) and Lioui (2005):
    - → Diffusion models with power utility.
    - → Lioui and Poncet (2001): last hedging component in terms of unknown volatility function (PDE).
    - → Lioui (2005): affine model with mean-reverting IR and MPR processes. Forward density hedge is proportional to vector of volatilities of MPR with proportionality factor linear in MPRs.

▶ Illustration: optimal stock-bond mix for CRRA investor

- Model:

→ $T-$maturity bond is traded

→ Two assets: Stock and investment horizon matching bond

$$\sigma_t = \begin{bmatrix} \sigma_{1t}^{stock} & \sigma_{2t}^{stock} \\ \sigma_{1t}^{B} & \sigma_{2t}^{B} \end{bmatrix}$$

- Optimal portfolio weight: static hedging component

$$\frac{\pi_t^b}{X_t^*} = \rho(\sigma_t')^{-1}\sigma_t^B = \rho \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- If $\pi_t^z \approx 0$ **hedging** is very simple:

→ no hedging component for stocks

→ hedging component does not depend on investment horizon

→ hedging portfolio only depends on relative risk aversion coefficient

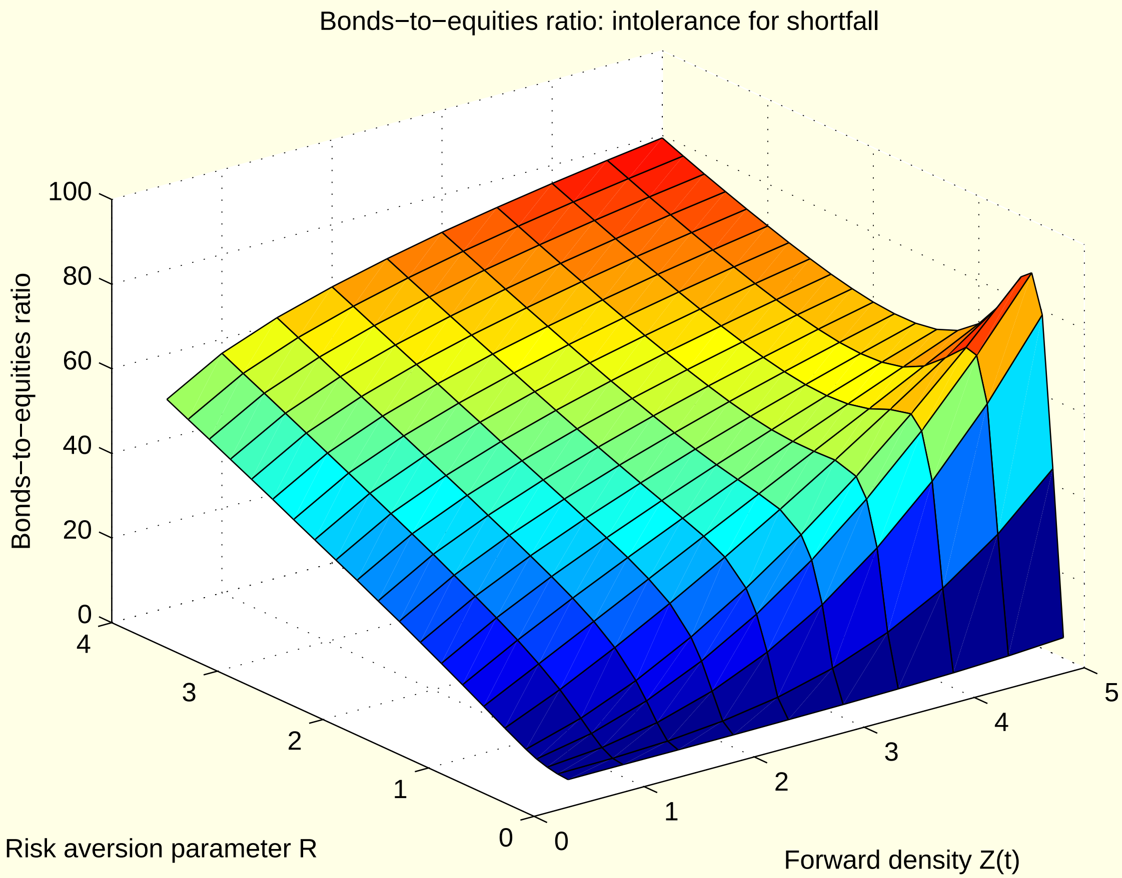
→ no need to estimate: if risk aversion is $R = 4$, then static hedging component for bonds is 0.75.

- Illustration: Asset Allocation Puzzle
  - Asset Allocation Puzzle (see Canner, Mankiw and Weil (1997)): investment advisors typically recommend an increase in the bonds-to-equities ratio for more conservative investors while mean-variance portfolio theory predicts that a constant ratio is optimal.
  - Bonds-to-equities ratio in Gaussian terms structure models:

$$e(t,T) = \sigma_t^S \frac{\theta_{2t}+\sigma_{2t}^B(Q(t,T)-1)}{\sigma_{2t}^B\theta_{1t}-\sigma_{1t}^B\theta_{2t}} \qquad Q(t,T) \equiv E_t^T\left[X_T^*\right]/E_t^T\left[\Gamma_T^*\right]$$

  - For HARA utility $u(x) = (x-A)^{1-R}/1-R$,
    - $Q(t,T;R) \equiv R\left(1+\left(\frac{B_0^T h(0,t;R)}{x/A-B_0^T}\right)\left(\frac{B_0^T/B_0^t}{B_t^T Z_t}\right)^{-1/R}\right) = R\left(1+\frac{AB_t^T}{X_t^*-AB_t^T}\left(B_0^T/B_0^t\right)^\rho\right)$
    - with
      - $h(0,t;R) \equiv \exp\left((\rho/R)\int_0^t \left(\frac{1}{2}\|\theta_s+\sigma^B(s,T)\|^2-\|\sigma^B(s,T)\|^2\right)ds\right)$.
  - Bonds-to-equities ratio risk tolerance, at a given time t, if and only if $Q(t,T)$ is a monotone function of risk tolerance.

- In the presence of wealth effect the bonds-to-equities ratio is not necessarily monotone in risk aversion



  – Vasicek interest rate model: $r_0 = \overline{r} = 0.06$, $\kappa_r = 0.05$, $\sigma_{r1} = -0.02$, $\sigma_{r2} = -0.015$ and market prices of risk are constants $\theta_s = 0.3$ and $\theta_b = 0.15$. The interest rate at $t = 5$ is $r_t = 0.02$. Other parameter values are $A = 200,000$, $x = 100,000$ and $T = 10$.

5 Intermediate Consumption

5.1 The Investor's Preferences

▶ Consumption-portfolio Problem:

$$\boxed{\max_{\pi,c\in\mathcal{A}} \mathbf{E}\left[\int_0^T u\left(c_t, t\right) dt + U(X_T)\right]}$$

- **Utility function:** $u\left(\cdot,\cdot\right) : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ and **bequest function:** $U : \mathbb{R}_+ \to \mathbb{R}$ satisfy standard assumptions
- Maximization over set of admissible portfolio policies $\pi, c \in \mathcal{A}$
- Inverse marginal utility function $J\left(y,t\right)$ exists: $u'\left(J\left(y,t\right),t\right) = y$ for all $t \in [0,T]$
- Inverse marginal bequest function $I\left(y\right)$ exists: $U'\left(I\left(y\right)\right) = y$

5.2 Portfolio Representation and Coupon-paying Bonds

- Decomposition:

$$\boxed{\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z}$$

- **Mean variance demand:**

$$\boxed{\pi_t^m = \left(\int_t^T E_t^v\left[\Gamma_v^*\right] B_t^v dv + E_t^T\left[\Gamma_T^*\right] B_t^T\right)\left(\sigma_t'\right)^{-1}\theta_t}$$

- **Hedge** motivated by fluctuations in price of **coupon-paying bond with matching maturity:**

$$\begin{aligned}\pi_t^b \quad = \quad & \left(\sigma_t'\right)^{-1}\int_t^T \sigma^B(t,v) B_t^v E_t^v\left[c_v^* - \Gamma_v^*\right] dv \\ & + \left(\sigma_t'\right)^{-1}\sigma^B(t,T) B_t^T E_t^T\left[X_T^* - \Gamma_T^*\right]\end{aligned}$$

- **Hedge** motivated by fluctuations in **density of forward T-measure**:

$$\begin{aligned}\pi_t^z \quad = \quad & \left(\sigma_t'\right)^{-1}\left(\int_t^T E_t^v\left[\left(c_v^* - \Gamma_v^*\right)\mathcal{D}_t \log Z_{t,v}\right] B_t^v dv\right)' \\ & + \left(\sigma_t'\right)^{-1}\left(E_t^T\left[\left(X_T^* - \Gamma_T^*\right)\mathcal{D}_t \log Z_{t,T}\right] B_t^T\right)'\end{aligned}$$

- Static Hedge π_t^b: hedge against fluctuations in value of coupon-paying bond
  - **Coupon payments** $\boxed{C(v) \equiv E_t^v [c_v^* - \Gamma_v^*]}$ at intermediate dates $v \in [0, T)$
  - **Bullet payment** $\boxed{F \equiv E_t^T [X_T^* - \Gamma_T^*]}$ at terminal date T
  - Coupon payments and face value are

    → time-varying

    → tailored to individual's consumption profile and risk tolerance
  - **Bond value**

$$\boxed{B(t, T; C, F) \equiv \int_t^T B_t^v C(v)\, dv + B_t^T F.}$$

  - **Instantaneous volatility**

$$\boxed{\begin{aligned} \sigma(B(t, T; C, F))\, B(t, T; C, F) &= \int_t^T \sigma^B(t, v)\, B_t^v C(v)\, dv \\ &\quad + \sigma^B(t, T)\, B_t^T F \end{aligned}}$$

  - **Hedge:** $\boxed{(\sigma_t')^{-1}\, \sigma(B(t, T; C, F))\ B(t, T; C, F)}$

- Forward Density Hedge π_t^z:
  - Motivation: desire to hedge **fluctuations in forward densities** $Z_{t,v}$
  - Static hedge already neutralizes impact of term structure fluctuations on PV of future consumption
  - Given $\xi_{t,v} = B_t^v Z_{t,v}$ it remains to hedge **fluctuations in risk-adjusted discount factors** $Z_{t,v}, v \in [t, T]$.
- Optimal Portfolio Composition:
  - To first approximation optimal portfolio has mean-variance term + long term coupon bond hedge
  - Under what conditions is this approximation exact (i.e. last term vanishes)?
  - If last term does not vanish what is its size?

5.3 Constant Relative Risk Aversion

- Relative risk aversion parameters R_u, R_U for utility and bequest functions. Portfolio:
  - Mean-variance term

$$\boxed{\pi_t^m = (\sigma_t')^{-1} \left(\int_t^T \frac{1}{R_u} E_t^v [c_v^*] B_t^v dv + \frac{1}{R_U} E_t^T [X_T^*] B_t^T \right) \theta_t}$$

  - Hedge motivated by fluctuations in price of coupon-paying bond with matching maturity

$$\boxed{\pi_t^b = (\sigma_t')^{-1} \left(\rho_u \int_t^T \sigma^B (t, v) B_t^v E_t^v [c_v^*] dv + \rho_U \sigma^B (t, T) B_t^T E_t^T [X_T^*] \right)}$$

  - Hedge motivated by fluctuations in densities of forward measures

$$\begin{aligned} \pi_t^z &= \rho_u (\sigma_t')^{-1} \int_t^T E_t^v [c_v^* \mathcal{D}_t \log Z_{t,v}]' B_t^v dv \\ &\quad + \rho_U (\sigma_t')^{-1} E_t^T [X_T^* \mathcal{D}_t \log Z_{t,T}]' B_t^T \end{aligned}$$

- Static Hedge has two parts:
  - Pure coupon bond (annuity) with coupon given by optimal consumption
  - Bullet payment given by optimal terminal wealth
  - Two parts are weighted by risk aversion factors ρ_u and ρ_U
  - Knife edge property traditionally associated with power utility function
  - Possibility of positive annuity hedge ($R_u > 1$) combined with negative bequest hedge ($R_U < 1$).

6 Applications

6.1 Preferred Habitats and Portfolio Choice

- Preferred Habitat Theory Modigliani and Sutch (1966):
  - Individuals exhibit preference for securities with maturities matching their investment horizon
  - Investor who cares about terminal wealth should invest in bonds with matching maturity
  - Existence of group of investors with common investment horizon might lead to increase in demand for bonds in this maturity range
  - Implies increase in bond prices and decrease in yields. Explains hump-shaped yield curves with decreasing profile at long maturities.

- Formula shows that optimal behavior naturally induces a demand for certain types of bonds in specific maturity ranges

$$\pi_t^* = w_t^m \left(X_t^* - B\left(t,T;C,F\right)\right) + w_t^b B\left(t,T;C,F\right) + \pi_t^z$$

$$w_t^m = \arg\max_w \left\{w'\sigma_t\theta_t : w'\sigma_t\sigma_t' w = k\right\}.$$

$$w_t^b = \arg\max_w \left\{w'\sigma_t\sigma\left(B\left(t,T;C\right)\right) : w'\sigma_t\sigma_t' w' = k\right\}$$

$$\pi_t^z = \arg\max_\pi \left\{\pi'\sigma_t\widehat{\sigma}\left(t,T\right) : \pi'\sigma_t\sigma_t'\pi' = k\right\}$$

where

$$\begin{aligned}\widehat{\sigma}_{t,T}' \quad \equiv \quad & \textstyle\int_t^T E_t^v\left[\left(c_v^* - \Gamma_v^*\right)\mathcal{D}_t \log Z_{t,v}\right] B_t^v dv \\ & + E_t^T\left[\left(X_T^* - \Gamma_T^*\right)\mathcal{D}_t \log Z_{t,T}\right] B_t^T\end{aligned}$$

- **Any individual has preferred bond habitat:**
  - → Optimal portfolio includes long term bond with maturity date matching the investor's horizon
  - → Preferred instrument is coupon-paying bond with payments tailored to consumption profile of investor
- **Complemented by mean-variance efficient portfolio to constitute the static component of allocation**
- Under general market conditions the static policy is fine-tuned by dynamic hedge
  - → When bond return volatilities and market prices of risk are deterministic, dynamic hedge vanishes

- Equilibrium Implications
  - Existence of natural preferred habitats in certain segments of fixed income market
  - Existence of equilibrium effects on prices and premia in these habitats depends on characteristics of investors' population
  - With sufficient homogeneity

  → Strong demand for structured fixed income products might emerge

  → Prompt financial institutions to offer tailored products appealing to those segments

  → Yields to maturity would then naturally reflect this habitat-motivated demand

- Motivation for preferred habitat here is different from Riedel (2001)
  - In his model habitat preferences are driven by structure of subjective discount rates placing emphasis on specific future dates
  - In our setting preference for long term bonds emerges from the structure of the hedging terms
  - Optimal hedging combines static hedge (long term bond) with dynamic hedge motivated by fluctuations in forward measure volatilities

6.2 Universal Fund Separation

- Y: vector of $N < d$ state variables with evolution described by the functional stoachastic differential equation

$$\boxed{dY_t = \mu(Y_{(\cdot)})_t dt + \sigma(Y_{(\cdot)})_t dW_t}$$

- Suppose that
  - $\boxed{B_t^v = B\left(t, v, Y_{(\cdot)}\right)}$
  - $\boxed{\sigma^Z(t,v) = \sigma^Z\left(t, v, Y_{(\cdot)}\right)}$

  are Fréchet differentiable functionals of $Y_{(\cdot)}$.

- : Universal $N+1$-fund separation holds: portfolio demands can be synthesized by investing in $N+2$ *(preference free) mutual funds*:
  1. riskless asset
  2. the mean-variance efficient portfolio
  3. N portfolios $(\sigma_t')^{-1}\sigma_t^Y\left(Y_{(\cdot)}\right)'$ to synthesize the static bond hedge and the forward density hedge.

6.3 Extreme Behavior

- Assume risk tolerances go to zero:
  - **Intermediate utility and bequest functions:**

$$\boxed{(\Gamma_u(z,v),\Gamma_U(z)) \rightarrow (0,0) \text{ for all } z \in [0,+\infty) \text{ and all } v \in [0,T]}$$

  - **Relative behaviors:** for some constant $k \in [0,+\infty)$:

$$\boxed{\frac{\Gamma_u(z_1,v)}{\Gamma_U(z_2)} \rightarrow k \quad \text{for all } z_1, z_2 \in [0,\infty) \ \text{ and all } v \in [0,T]}$$

$$\boxed{\frac{\Gamma_u(z_1,v_1)}{\Gamma_u(z_2,v_2)} \rightarrow 1 \quad \text{for all } z_1, z_2 \in [0,\infty) \ \text{ and all } v_1, v_2 \in [0,T]}$$

- Limit Allocations: coupon-paying bond with constant coupon C and face value F given by

$$\boxed{C = \frac{x}{\int_0^T B_0^v dv + B_0^T/k} \qquad \text{and} \qquad F = \frac{x}{\int_0^T B_0^v dv k + B_0^T}.}$$

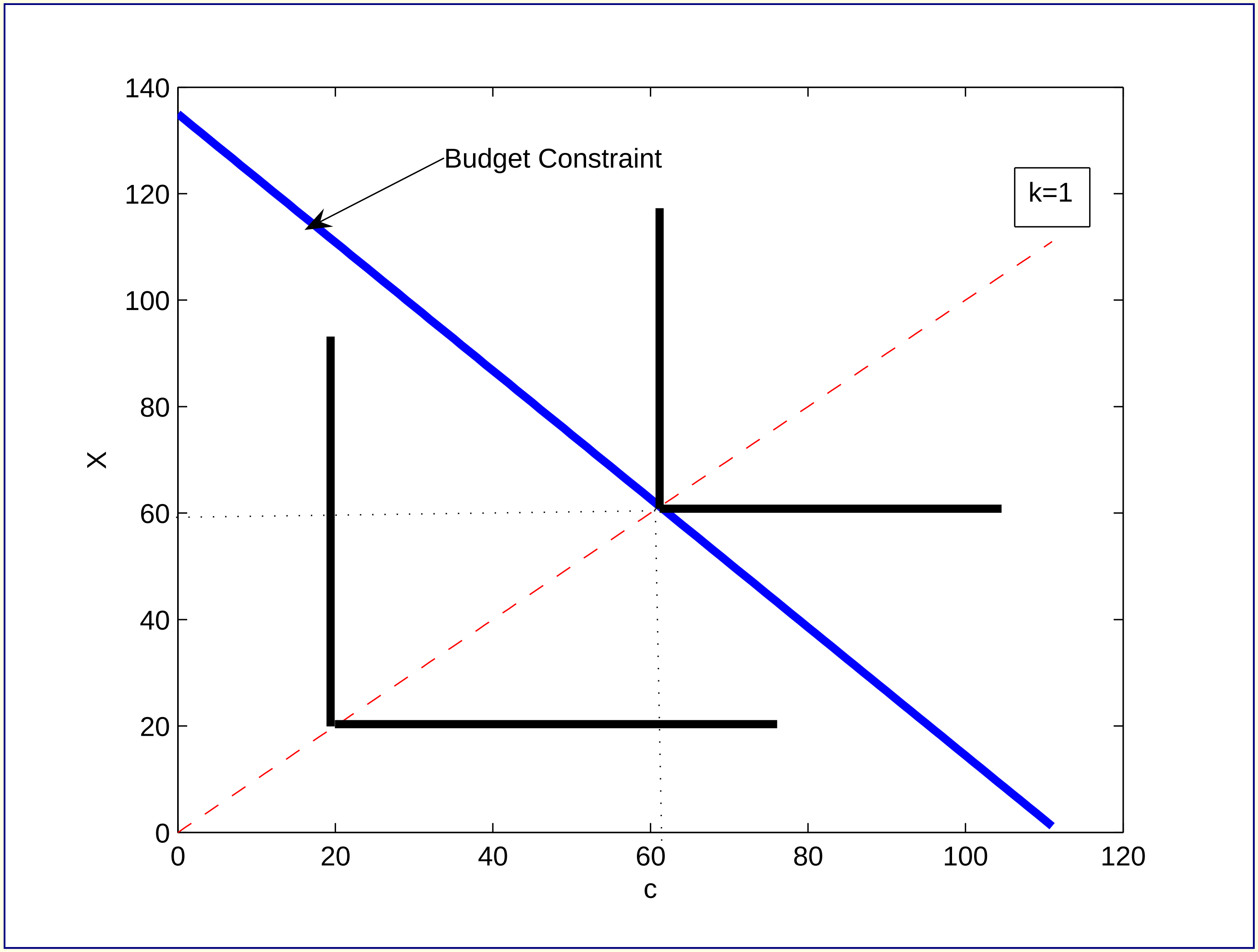
  - If $k = 0$ exclusive preference for pure discount bond, $(C, F) = \left(0, x/B_0^T\right)$
  - If $k \to \infty$ preference is for a pure coupon bond, $(C, F) = \left(x/\int_0^T B_0^v dv, 0\right)$

- Limit Behavior:
  - Governed by relation between utility functions at different dates
  - As risk tolerances vanish, preference for certainty: coupon-paying bond with bullet payment
  - Least extreme of the extreme behaviors drives the habitat:

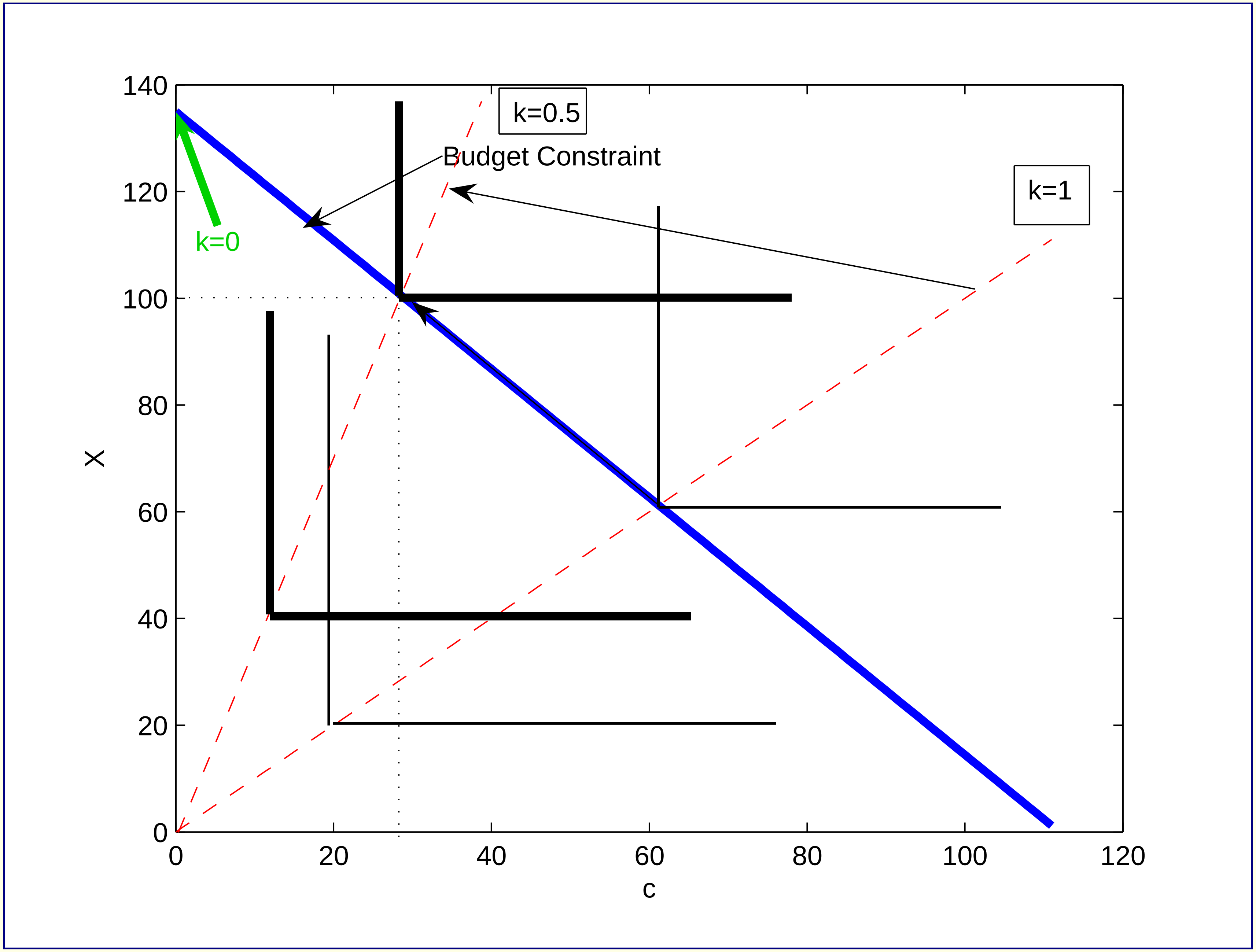
  → Given a preference for riskless instruments: individuals puts more weight on maturities where risk tolerance is greater

  → Exhibits a time preference in the limit.

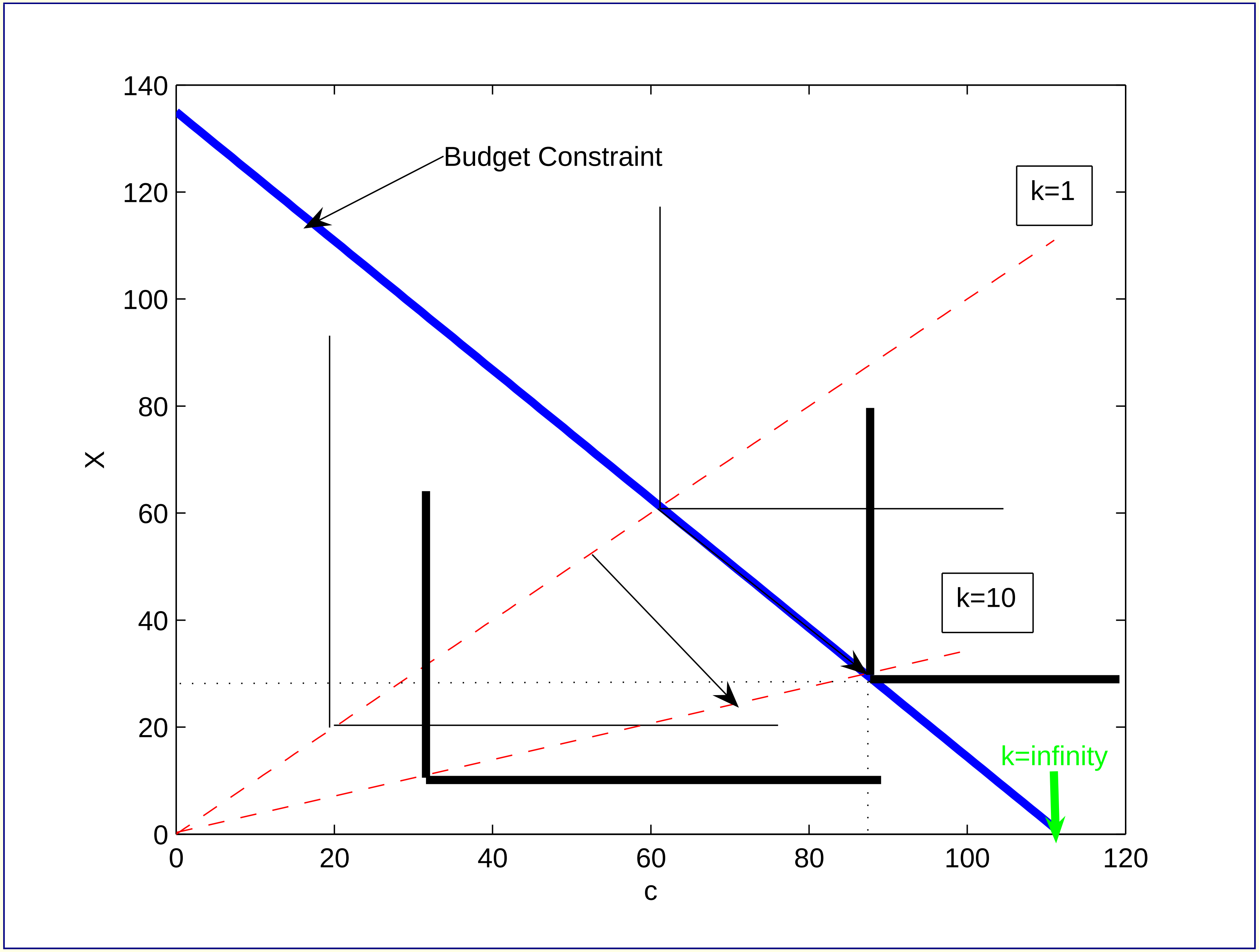
- Illustration: CARA preferences Γ_u and Γ_U constant, $k \equiv \Gamma_u/\Gamma_U$.
  - Slope of indifference curves: $$-\frac{dX}{dc} = \frac{1}{k}\left(e^{X-c/k}\right)^{\frac{1}{\Gamma_U}}$$



- $k \to 0$



- $k \rightarrow \infty$



- Special case examined by Wachter (2002)
  - Arbitrary utility functions over terminal wealth and markets with general coefficients
  - Documents emergence of preferred habitat when relative risk aversion goes to infinity
    - → Pure discount bond with unit face value and matching maturity
  - Our analysis shows that preferred habitat for an extreme consumer may take different forms depending on nature of behavior
    - → Pure discount bonds, pure annuities or coupon-paying bonds with bullet payments at maturity can emerge in limit.

► Order of Convergence

- As $(\Gamma_u(z,v), \Gamma_U(z)) \to (0,0)$, the limit portfolios
  * $\overline{\pi}_t^m = \overline{\pi}_t^z = 0$
  * $\overline{\pi}_t^b = (\sigma_t') \int_0^T \sigma^B(t,v) B_t^v dv C + \sigma^B(t,T) B_t^T F$
- have scaled asymptotic errors:
  * $\epsilon_t^\alpha(\nu) = (\Gamma_\nu(\cdot))^{-1} (\pi_t^\alpha - \overline{\pi}_t^\alpha)$ with $\alpha \in \{m,b,z\}$ and $\nu \in \{u,U\}$,

$$\begin{array}{rcc} [\epsilon_t^m(U), \epsilon_t^m(u)] & \to & (\sigma_t')^{-1} \theta_t \left[\int_t^T B_t^v dv \; B_t^T \right] \mathcal{K} \\ [\epsilon_t^b(U), \epsilon_t^b(u)] & \to & -(\sigma_t')^{-1} \left[\int_t^T \sigma^B(t,v) B_t^v dv \; \sigma^B(t,T) B_t^T \right] \mathcal{K} \\ [\epsilon_t^z(U), \epsilon_t^z(u)] & \to & -(\sigma_t')^{-1} \left[\int_t^T N_{t,v} B_t^v dv \; N_{t,T} B_t^T \right] \mathcal{K} \end{array}$$

- where
  * $N_{t,\tau}$ is given by

$$N_{t,\tau} \equiv E_t^\tau \left[\left(\int_t^\tau \sigma^Z(r,\tau)' dW_r - \frac{1}{2} \int_t^\tau \|\sigma^Z(r,\tau)\|^2 dr \right) (\mathcal{D}_t \log Z_{t,\tau})' \right]$$

  * $\mathcal{K}$ is given by

$$\mathcal{K} \equiv \left[\begin{array}{cc} k & 1 \\ 1 & \frac{1}{k} \end{array} \right]$$

.

6.4 Term structure models and asset allocation

- Integration of term structure models and asset allocation models:

  - Forward rate representation of bonds

$$\boxed{B_t^v = \exp\left(-\int_t^v f_t^s ds\right)}$$

→ Continuously compounded forward rate: $f_t^s \equiv -\frac{\partial}{\partial v} \log\left(B_t^v\right)$

→ Bond price volatility:

$$\boxed{\sigma^B(t,v)' = \mathcal{D}_t \log B_t^v = -\int_t^v \mathcal{D}_t f_t^s ds = -\int_t^v \sigma^f(t,s) ds}$$

→ Volatility of forward rate: $\sigma^f(t,s)$

  - Forward rate dynamics:

→ No arbitrage condition (HJM (1992)):

$$\boxed{df_t^v = \sigma^f(t,v)\left(dW_t + \left(\theta_t - \sigma^B(t,v)\right)dt\right), \quad f_0^v \text{ given}}$$

→ Dynamics completely determined by forward rate volatility function and initial forward rate curve

- Optimal Portfolio: previous formula with

$$\mathcal{D}_t \log Z_{t,v} = \int_t^v \left(dW_s + \left(\theta_s + \int_s^v \sigma^f(s,u)du\right) ds\right)' \left(\mathcal{D}_t\theta_s + \int_s^v \mathcal{D}_t\sigma^f(s,u)du\right)$$

  - Forward density hedge in terms of forward rate volatilities
  - Useful for financial institution using a specific HJM model to price/hedge fixed income instruments and their derivatives
  - Implied forward rates inferred from term structure model and observed prices

    → estimate volatility function $\sigma^f(s,u)$

    → feed into asset allocation formula
  - Simple integration of fixed income management and asset allocation.

- **Forward Density Hedge:**
  - Immunization demand due to fluctuations in future market prices of risk and forward rate volatilities
  - Vanishes if deterministic forward rate volatilities $\sigma^f(s,u)$ and market prices of risk θ_s
  - Pure expectation hypothesis holds under forward measure: $f(t,v) = E_t^v[r_v]$

  → Standard version of PEH ($f(t,v) = E_t[r_v]$) fails when $Z_{t,v} \neq 1$

  → Density process $Z_{t,v}$ measures deviation from PEH

  → Malliavin derivative $\mathcal{D}_t \log Z_{t,v}$ captures sensitivity of deviation with respect to shocks

  → Dynamic hedge = hedge against deviations from PEH

  → If $Z_{t,v} = 1$ PEH holds under the original beliefs and hedging becomes irrelevant

  → If σ^Z deterministic, deviations from PEH are non-predictable and do not need to be hedged

- Literature:
  - Gaussian models: Merton (1974), Vasicek (1977), Hull and White (1990), Brace, Gatarek and Musiela (1997)
  - Extensively employed in practice
  - Forward rate volatilities σ^f are insensitive to shocks. If MPR also deterministic no need to hedge
  - Bajeux-Besnainou, Jordan and Portait (2001) also falls in this category (one factor Vasicek)

- Numerical Results: Forward measure hedges in one factor CIR model

  - CIR interest rates:

$$dr_t = \kappa_r(\bar{r} - r_t))dt + \sigma_r \sqrt{r} dW_t; \quad r_0 = r$$

  → Parameter values (Durham (JFE, 2003)):

  - $\kappa_r = 0.002$
  - $\bar{r} = 0.0497$
  - $\sigma_r = -0.0062$
  - $r = 0.06$

  - Market price of risk:

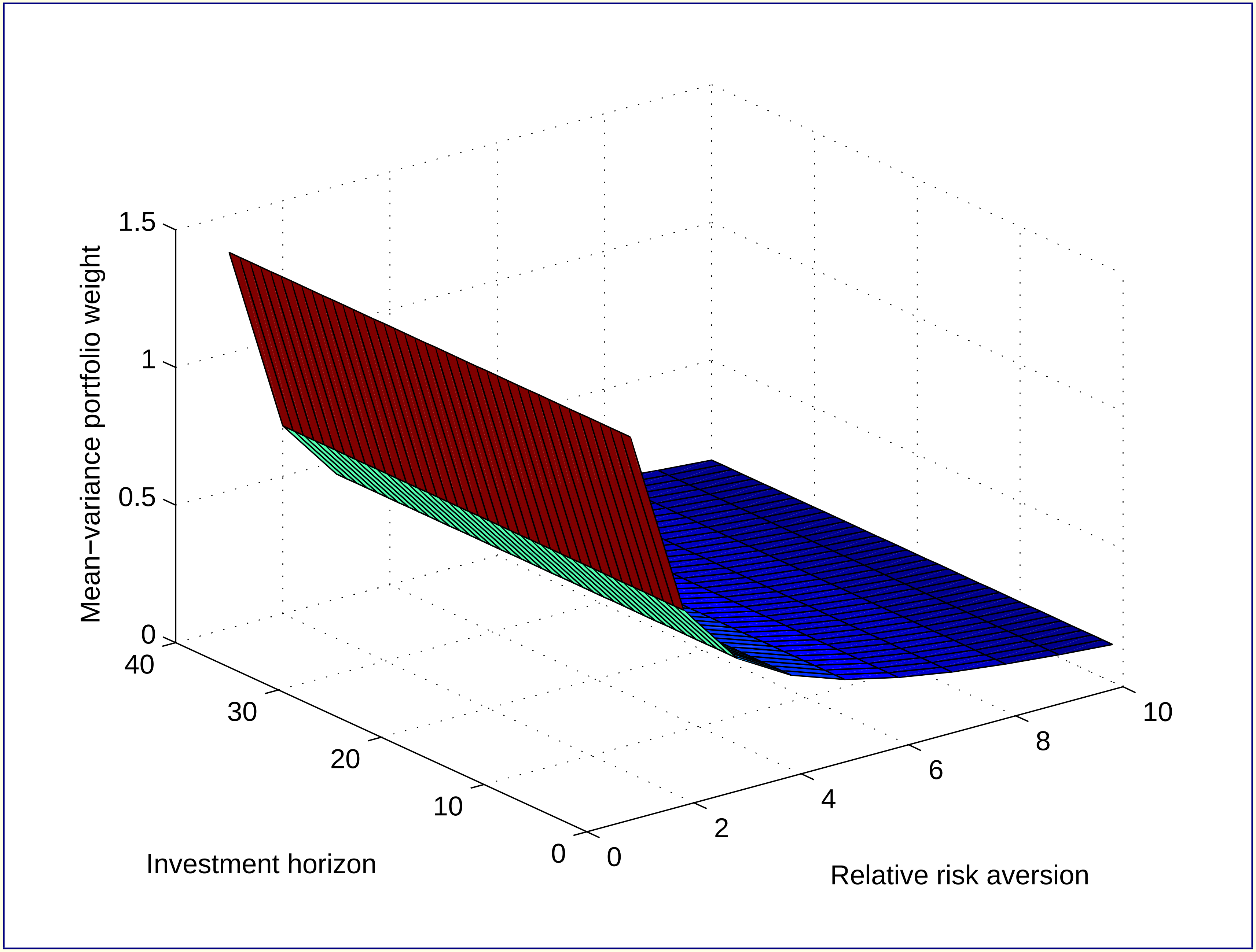
$$\theta_t = \gamma_r \sqrt{r_t}$$

  → Parameter values:

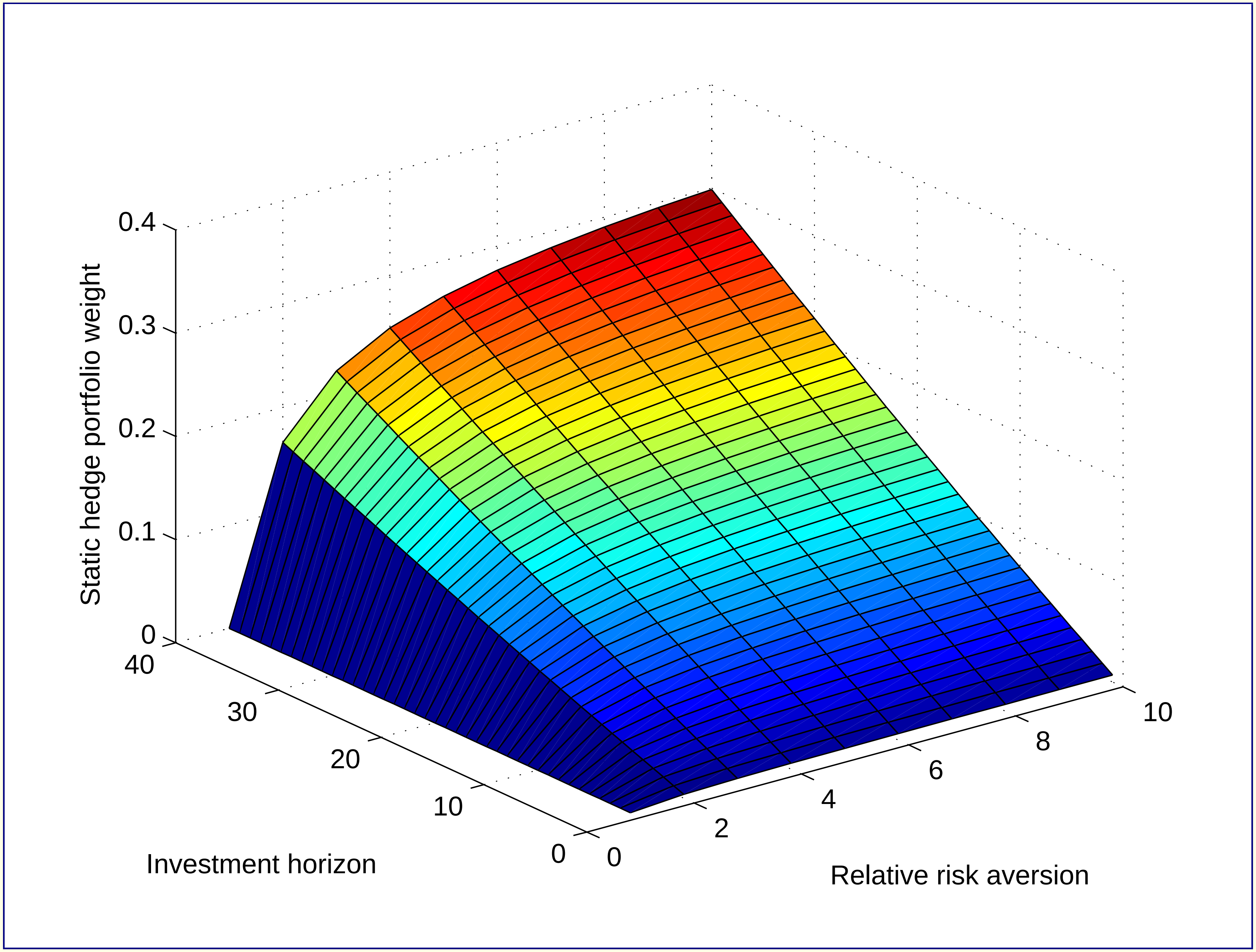
  - $\gamma_r = 0.3/\sqrt{\bar{r}}$ such that $\bar{\theta} = \gamma_r \sqrt{\bar{r}} = 0.3$

  - CRRA preferences for terminal wealth

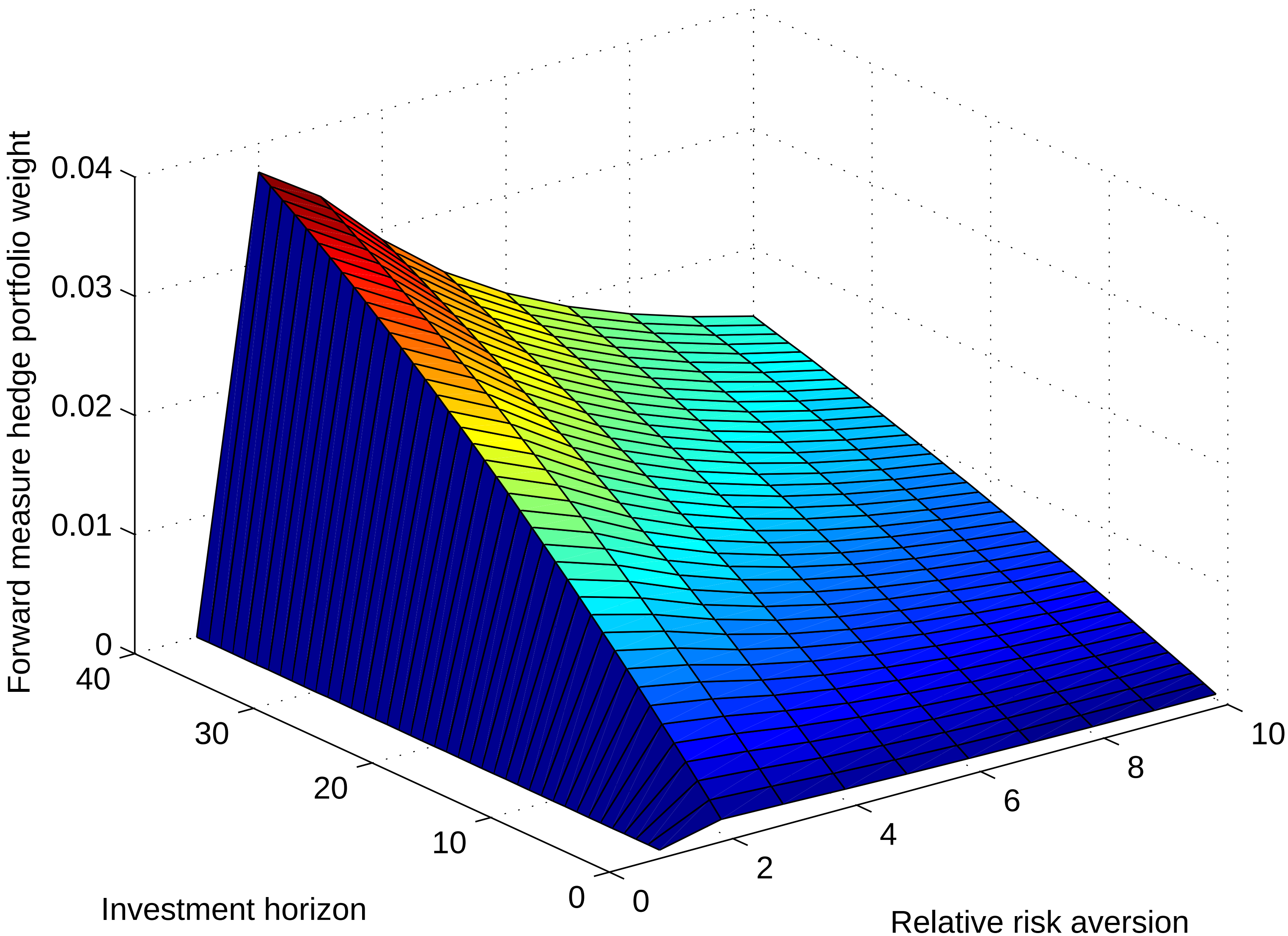
- Mean-variance demand: $\boxed{\pi_t^{mv}/X_t^* = \frac{1}{R}(\sigma_t')^{-1}\theta_t}$



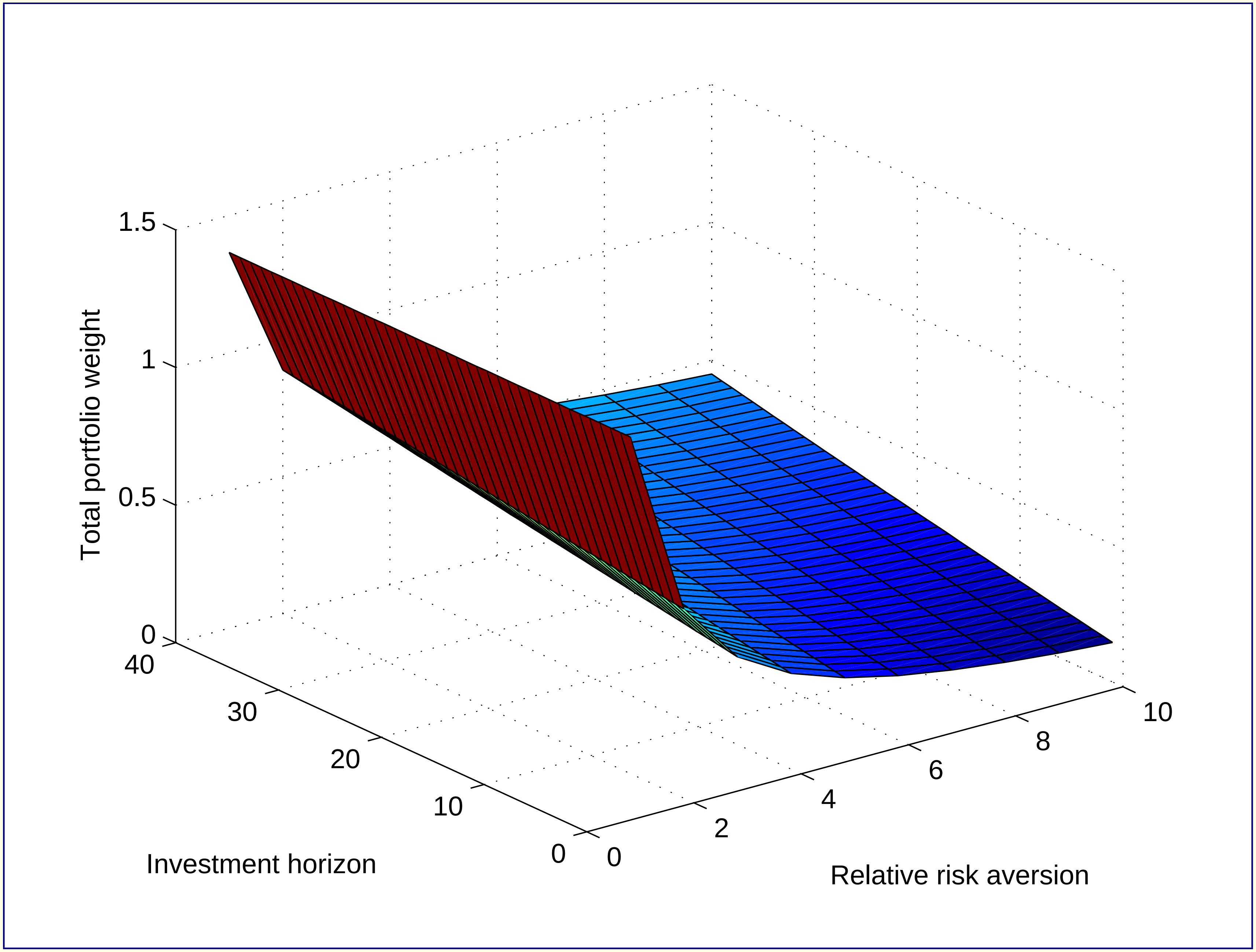
- Static term structure hedge: $\boxed{\pi_t^b / X_t^* = \rho(\sigma_t')^{-1} \sigma^B(t,T)}$



- Dynamic forward measure hedge: $\pi_t^z / X_t^* = \rho \left(\sigma_t'\right)^{-1} \mathbf{E}_t^T \left[\frac{Z_{t,T}^{\rho-1}}{\mathbf{E}_t^T \left[Z_{t,T}^{\rho-1} \right]} \left(\mathcal{D}_t \log Z_{t,T} \right)' \right]$

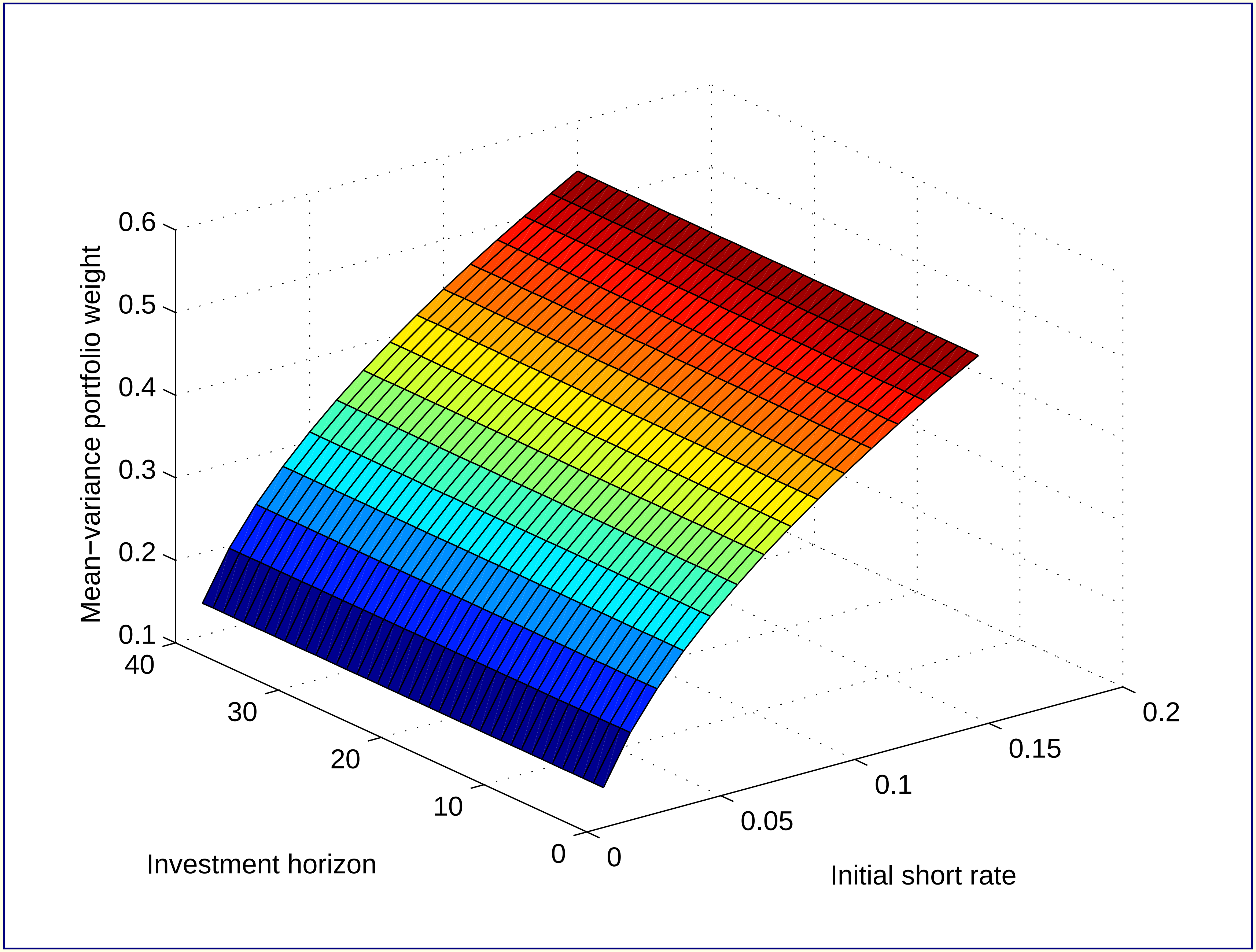


- Total portfolio weight: $\pi_t/X_t^* = \pi_t^{mv}/X_t^* + \pi_t^b/X_t^* + \pi_t^z/X_t^*$

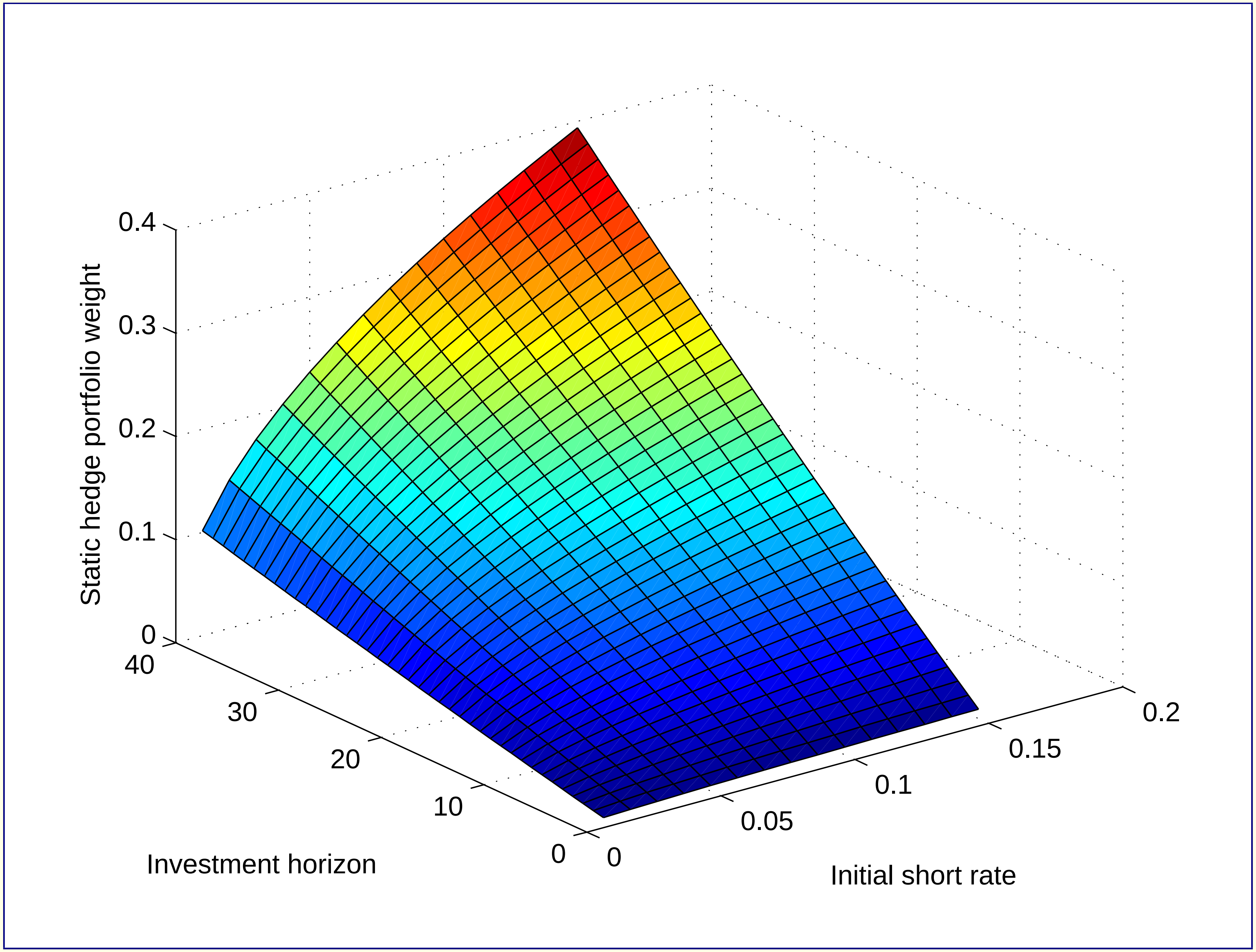


- Changing initial interest rate: Relative risk aversion fixed at $R = 4$

$\rightarrow$ Mean-variance demand: $\boxed{\pi_t^{mv}/X_t^* = \frac{1}{R}(\sigma_t')^{-1}\theta_t}$

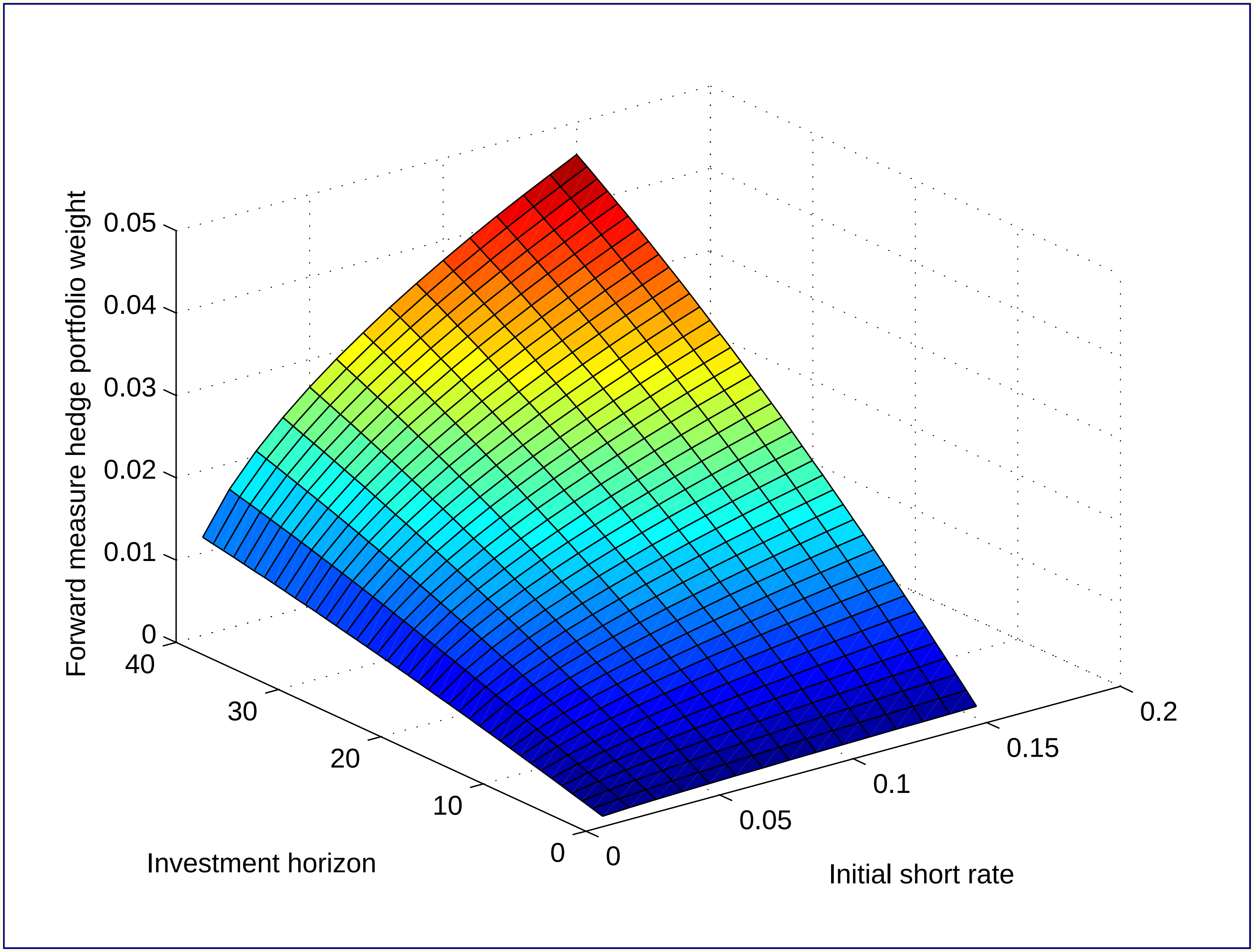


→ Static term structure hedge: $\boxed{\pi_t^b / X_t^* = \rho(\sigma_t')^{-1}\sigma^B(t,T)}$

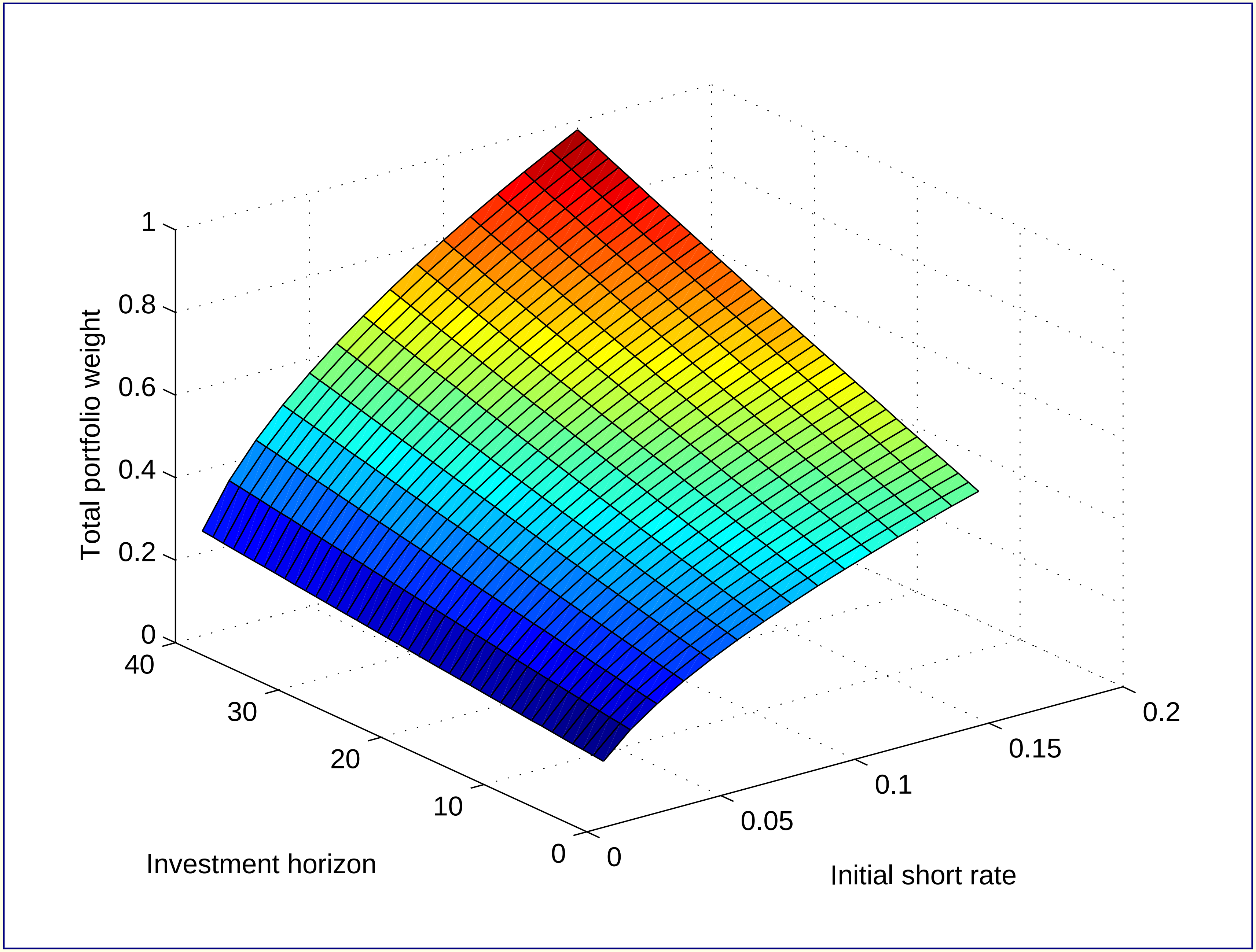


- Dynamic forward measure

hedge: $\pi_t^z / X_t^* = \rho \left(\sigma_t'\right)^{-1} \mathbf{E}_t^T \left[\frac{Z_{t,T}^{\rho-1}}{\mathbf{E}_t^T \left[Z_{t,T}^{\rho-1} \right]} \left(\mathcal{D}_t \log Z_{t,T} \right)' \right]$

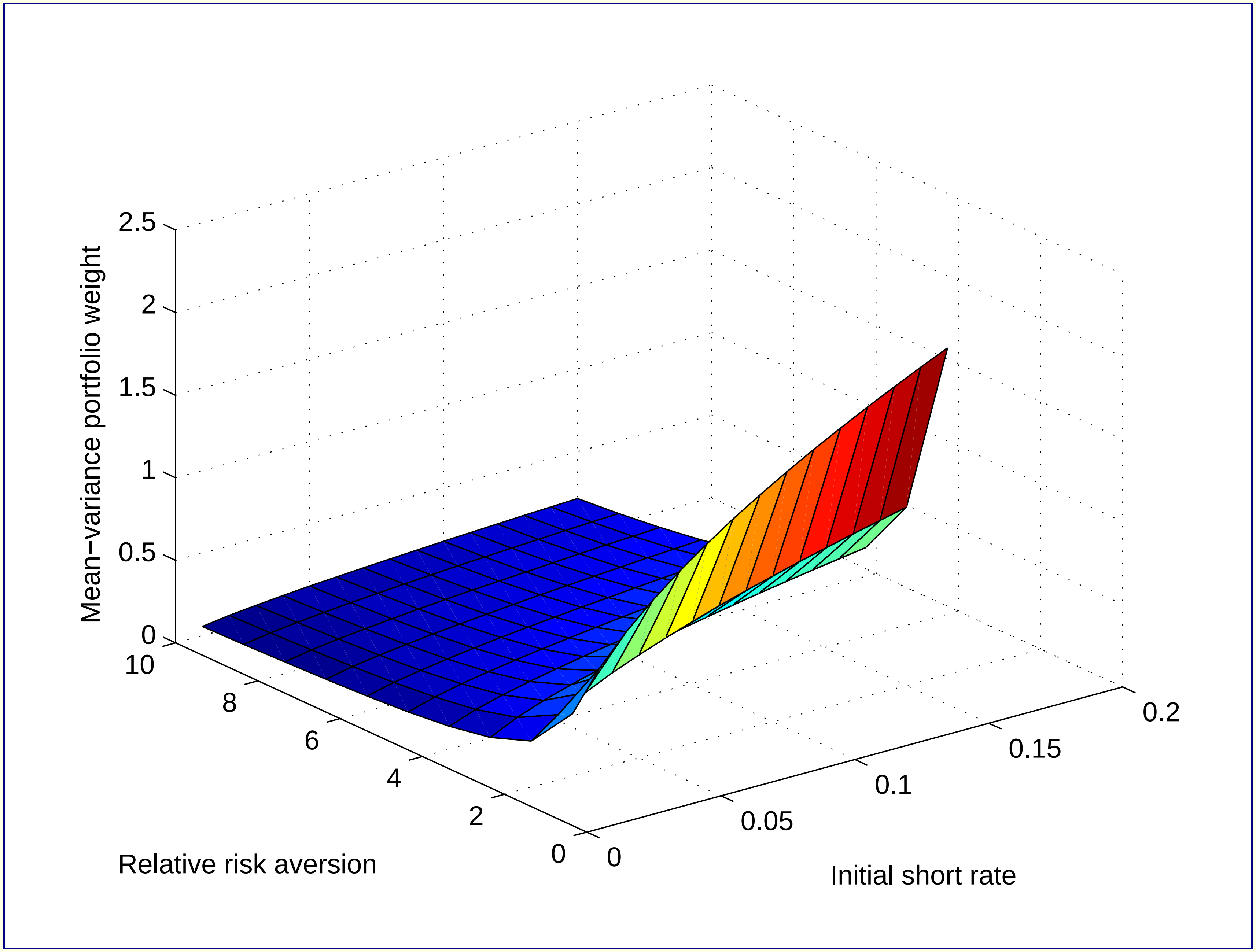


- Total portfolio weight: $\boxed{\pi_t/X_t^* = \pi_t^{mv}/X_t^* + \pi_t^b/X_t^* + \pi_t^z/X_t^*}$

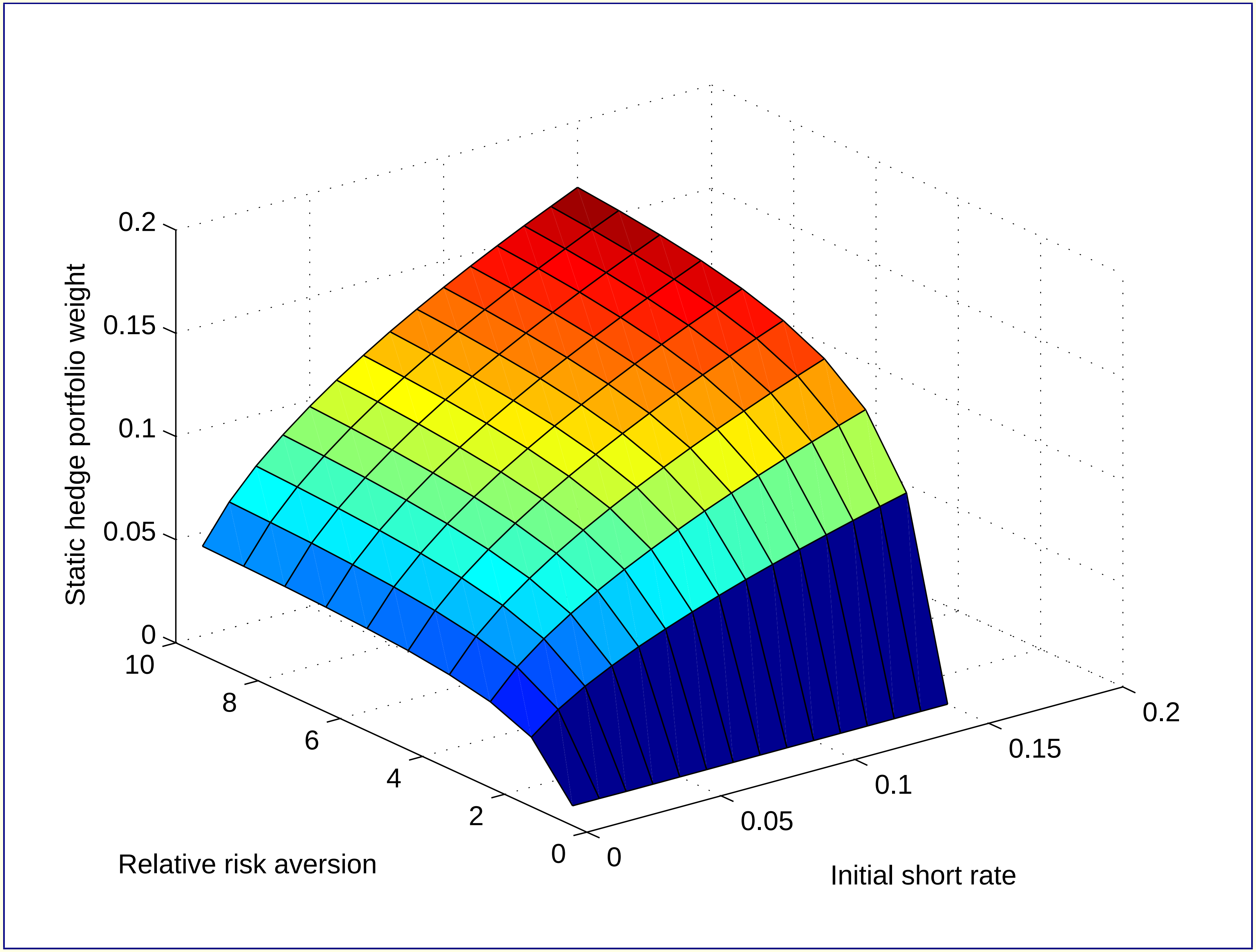


- Changing initial interest rate: Investment horizon fixed at $T = 15$

$\rightarrow$ Mean-variance demand: $\boxed{\pi_t^{mv}/X_t^{ast} = \frac{1}{R}(\sigma_t')^{-1}\theta_t}$

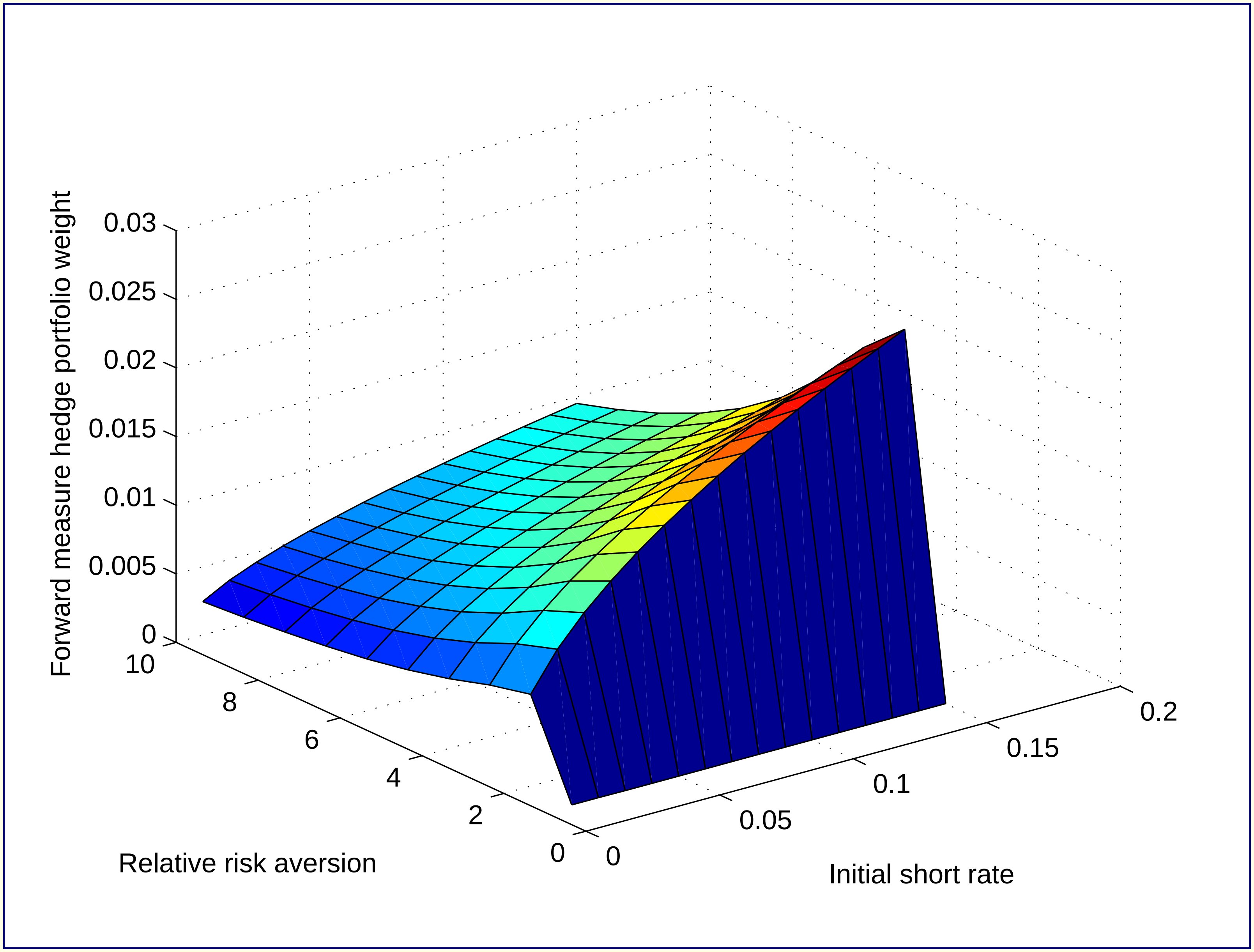


→ Static term structure hedge: $\boxed{\pi_t^b / X_t^* = \rho(\sigma_t')^{-1}\sigma^B(t,T)}$

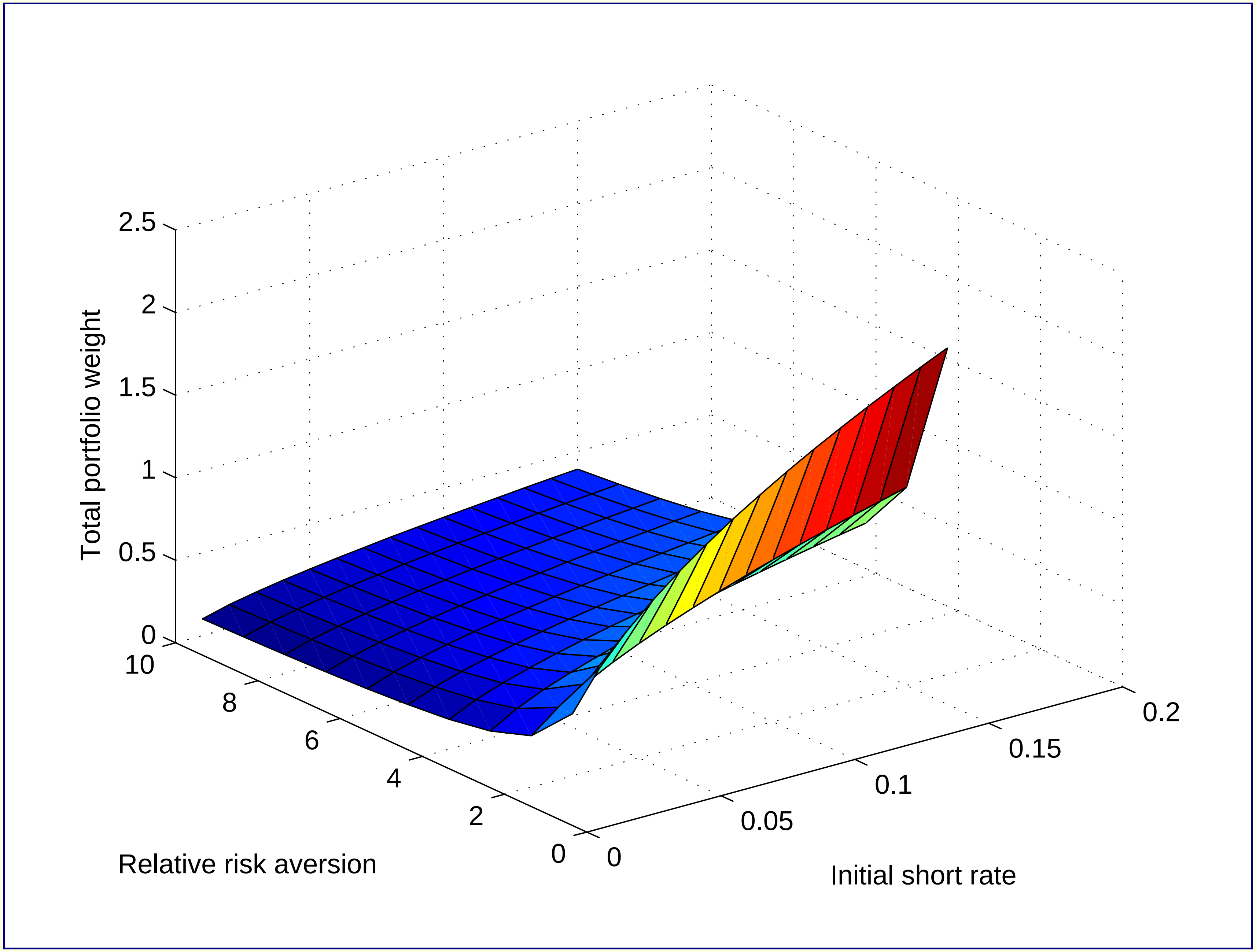


- Dynamic forward measure

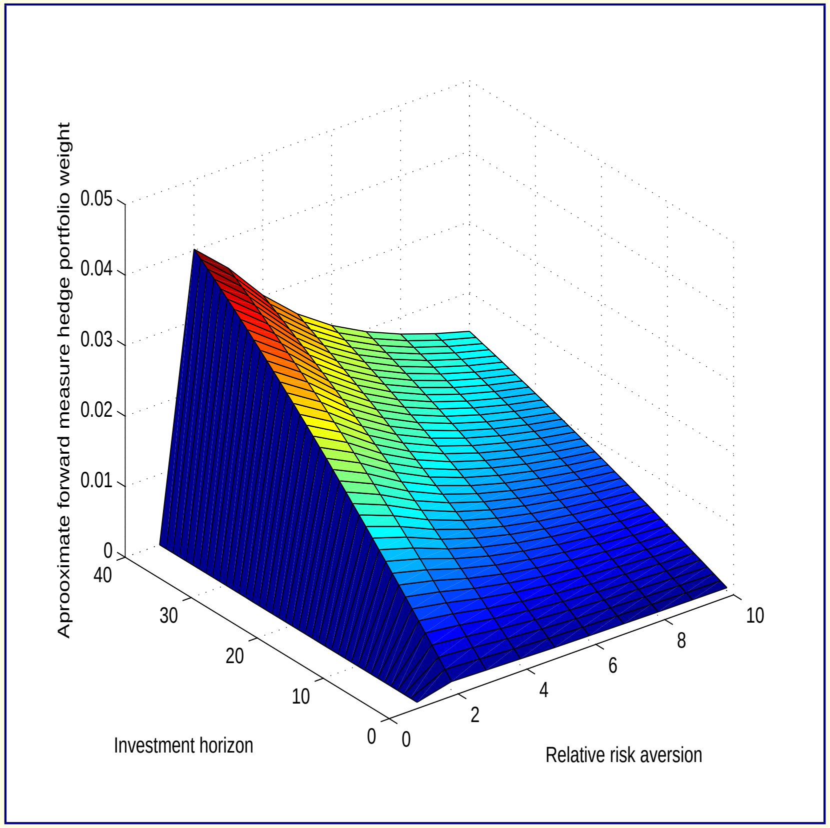
hedge: $\pi_t^z / X_t^* = \rho\,(\sigma_t')^{-1}\,\mathbf{E}_t^T\left[\frac{Z_{t,T}^{\rho-1}}{\mathbf{E}_t^T\left[Z_{t,T}^{\rho-1}\right]}\,(\mathcal{D}_t \log Z_{t,T})'\right]$



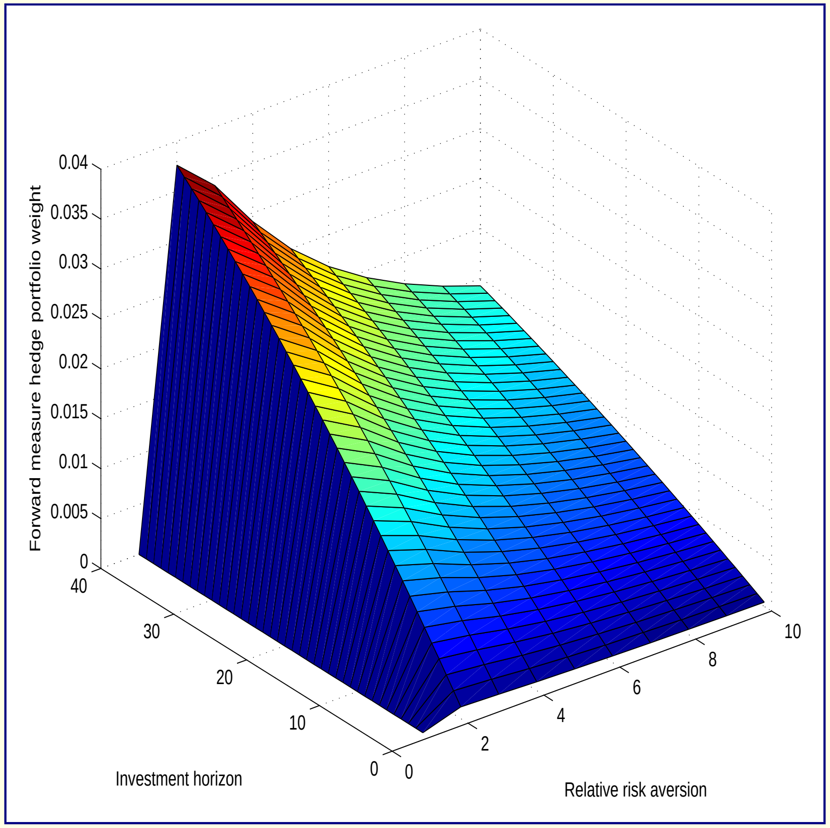
- Total portfolio weight: $\pi_t/X_t^* = \pi_t^{mv}/X_t^* + \pi_t^b/X_t^* + \pi_t^z/X_t^*$



- Approximate forward density portfolio weight: $\boxed{\pi_t^{fa}/X_t^* = -\sigma_t^{-1}\frac{1}{R}E_t^T[N_{t,T}]}$

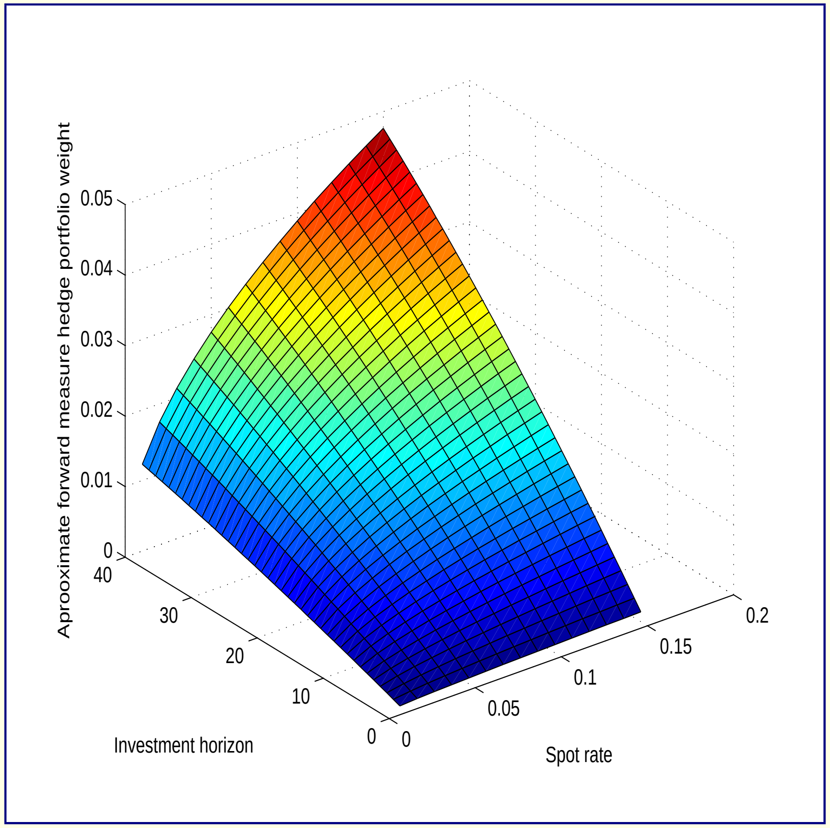


Approximate

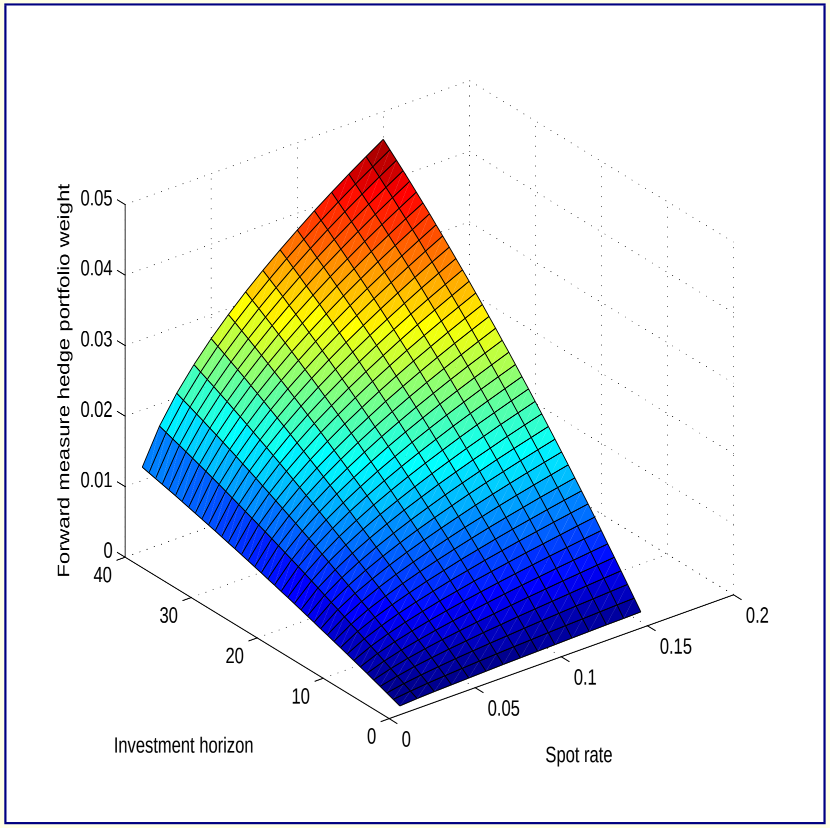


True

- Approximate forward density portfolio weight: $\pi_t^{fa}/X_t^* = -\sigma_t^{-1}\frac{1}{R}E_t^T[N_{t,T}]$



Approximate



True

7 Conclusion

- Contributions:
  - Asset allocation formula based on change of numéraire
  - Highlights role of consumption-specific coupon bonds as instruments to hedge fluctuations in opportunity set
  - Formula has multiple applications: preferred habitat, extreme behavior, international asset allocation, demand for I-bonds
  - Exponential Clark-Haussmann-Ocone formula
  - Malliavin derivatives of functional SDEs
  - Solution of linear BVIE
- Integration of portfolio management and term structure models
  - Asset allocation in HJM framework
  - Other applications
- Universal $N+2$ fund separation result