Filtration Shrinkage and Credit Risk
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Philip Protter, Cornell University

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Credit Risk

Credit risk investigates an entity (corporation, bank, individual) that borrows funds, promises to return them in a specified contractual manner, and who may not do so (default).

Mathematical framework: \((\Omega, \mathcal{G}, P, \mathcal{G})\) are given, 
\[ \mathcal{G} = (\mathcal{G}_t)_{t \geq 0}, 0 \leq t \leq T, \] usual hypotheses. For sake of this talk, let us consider a firm. The asset value process is

\[
dA_t = A_t \alpha(t, A_t) dt + A_t \sigma(t, A_t) dW_t
\]

where \(\alpha\) and \(\sigma\) are such that \(A\) exists, is well defined, and positive.

Assume the liability structure of the firm is a single zero-coupon bond with maturity \(T\) and face value 1, and default occurs only at time \(T\), and only if \(A_T \leq 1\).
• The probability of default is $P(A_T \leq 1)$. 
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• Time zero value of the firm’s debt is

$$v(0, T) = E_Q((A_T \land 1 \exp(-\int_0^T r_s ds)))$$.

This is the Black-Scholes-Merton model (from the early 1970s) viewed as a European call option on the firm’s assets, maturity $T$, and strike price equal to the value of the debt.
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• The Black-Scholes-Merton model has been extended to default before time \( T \) by considering a barrier \( L = (L_t)_{t \leq 0} \). Augment the information to include the \( L \) information. Let \( \mathcal{H}_t = \sigma(A_s, L_s; s \leq t) \).
• Default time becomes a first passage time relative to the barrier $L$:

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• The value of the firm’s debt is

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v(0, T) = E \left[ \left\{ 1_{\{t \leq T\}} L_\tau + 1_{\{\tau > T\}} 1 \right\} \exp(- \int_0^T r_s ds) \right].
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• The previous models are known as the **structural approach** to credit risk. The default time is predictable.
• Alternative: the **reduced form** approach of Jarrow–Turnbull, and Duffie–Singleton, of the 1990s. The observer sees only the filtration generated by the default time $\tau$ and a vector of state variables $X_t$.

\[ F_t = \sigma(\tau \wedge s, X_s; s \leq t) \subset G_t \]
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• Assume a trading economy with a risky firm with outstanding debt as zero coupon bonds. Assuming no arbitrage (but not completeness), there is an equivalent local martingale measure \( \mathcal{Q} \).

• Let

\[
M_t = 1_{\{t \geq \tau\}} - \int_0^t \lambda_s ds = a \quad \mathcal{Q} \quad \mathcal{F} - \text{martingale.}
\]

**Recovery rate** given by \((\delta_t)_{0 \leq t \leq T}\); So change \( \mathcal{F} \):

\[
\mathcal{F}_t = \sigma(\tau \wedge s, X_s, \delta_s; s \leq t) \subset \mathcal{G}_t
\]

\[
\mathcal{F}^X_t = \sigma(X_s; s \leq t).
\]
Default prior to time $T$:

$$Q(\tau \leq T) = E_Q \left\{ E_Q(N_T = 1|\mathcal{F}^X) \right\}$$

$$= E_Q(\exp(-\int_0^T \lambda_s ds)),$$

and value of the firm’s debt is

$$\nu(0, T) = E \left\{ [1_{\{\tau \leq T\}} \delta_\tau + 1_{\{\tau > T\}} 1] \exp(-\int_0^T r_s ds) \right\}$$

The modeler does not see the process $A = (A_t)_{t \geq 0}$, but has instead only partial information. How does one model this partial information?
There are three main approaches, in general:

1. **Duffie-Lando, Kusuoka**: Observe $A$ only at discrete intervals, and add independent noise.

2. With Kusuoka, a twist is given by introduction of filtration expansion.

3. **Giesecke-Goldberg**: The default barrier is a *random* curve; but $A$ is still assumed to observed continuously.

4. **Çetin-Jarrow-Protter-Yildirim**: Begin with a structural model under $\mathbb{G}$, and then project onto smaller filtration $\mathbb{F}$; Use of cash flows.
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$X$ denotes the cash balance of the firm, normalized by the money market account

$$dX_t = \sigma dW_t, \quad X_0 = x,$$

where $x > 0, \sigma > 0$

$\mathcal{Z} = \{ t \in [0, T] : X_t = 0 \}$

$g_t = \sup\{s \leq t : X_s = 0; g_t \text{ is the last time before } t \text{ cash balance is zero}\}$

$\tau_{\alpha} = \inf\{t > 0 : t - g_t \geq \frac{\alpha^2}{2} : X_s < 0, \text{ all } s \in (g_{t-}, t)\}$

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$$\tau = \inf\{t > \tau_\alpha : X_t = 2X_{\tau_\alpha}\}$$
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• But, the investor does not see the entire cash balance process.
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- $G$ is the filtration of $\text{sign}(Y_t)$
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$A_t = \int_0^{t \wedge \tau} \lambda_s \, ds$, and moreover $\lambda_t = 1_{\{t > \tau \text{g.a.}\}} \frac{1}{2(t - \tilde{g}_t)}, 0 \leq t \leq \tau$, and where $\tilde{g}_t = \sup\{s \leq t : Y_s = 0\}$. 

Nota Bene: We are able to calculate $\lambda_t$ explicitly since we have a formula for the Azéma martingale. Use knowledge of $\lambda_t$ to calculate quantities of interest. Simple example: price of a risky zero coupon bond at time 0: $S_0 = \exp\left(-\int_0^T r_u \, du\right) \left\{1 - \left(Q(\tau \alpha \leq T) - E(\alpha/\sqrt{2} \sqrt{T - \tilde{g}_\tau \alpha} \cdot 1_{\{\tau \alpha \leq T\}})\right)\right\}$. 
\[ N_t = 1_{t \geq \tau} \] with \( \mathbb{G} \) compensator \( A \)

**Theorem**
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A_t = \int_0^{t \land \tau} \lambda_s ds, \quad \text{and moreover} \quad \lambda_t = 1_{t > \tau^g a} \frac{1}{2(t - \tilde{g}_t)}, \quad 0 \leq t \leq \tau,
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and where \( \tilde{g}_t = \sup\{s \leq t : Y_s = 0\} \).

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• **Nota Bene:** We are able to calculate $\lambda$ explicitly since we have a formula for the Azéma martingale.

• Use knowledge of $\lambda$ to calculate quantities of interest. Simple example: price of a risky zero coupon bond at time $0$:

$$S_0 = \exp(-\int_0^T r_u du) \left\{1 - \left( Q(\tau_\alpha \leq T) - E\left(\frac{\alpha/\sqrt{2}}{\sqrt{T - \tilde{g}_{\tau_\alpha}}} 1_{\{\tau_\alpha \leq T\}}\right)\right)\right\}$$
Default occurs not at time $\tau_\alpha$, but at time $\tau$. The default time $\tau$ is, therefore, less likely than the hitting time $\tau_\alpha$. The probability $\mathbb{Q} [\tau_\alpha \leq T]$ is reduced to account for this difference.
• The preceding is both artificial and simple. Let us consider a more realistic situation. Use of Markov process theory and homogeneous regenerative sets (theory of Mémin and Jacod).
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• Instead of just using when the cash flow is positive or negative, we can look at when it crosses a grid of barriers. And instead of looking at just Brownian motion as cash flows, we can consider a diffusion $X$. 

• $F$ denotes the information from the crossings. $g_t$ denotes the last exit time before $t$ that $X$ crosses a level set in our collection. $U_t = t - g_t$ is the since since last exit. $F$ can be thought of as generated by $(X_{g_t}, \text{sign}(X_t - X_{g_t}) U_t)_{t \geq 0}$. 
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We discuss upward and downward excursions, and where they end up. For each type of excursion there corresponds a Lévy measure on $(0, \infty]$, which we denote by $F_i^{j\pm}$

$A^+ (\text{resp. } A^-)$ is for an upward (resp. downward) excursion

$j = 0 (\text{resp. } 1)$ is for an excursion ending at $x_i$ (resp. $x_{i \pm 1}$).

These measures are constructed using the excursion measure $n_i$ of $X$ at $x_i$. 
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•

Theorem

\(P\) almost surely, for all \(0 < t < \tau\),

\[
A_t = \begin{cases} 
A_{gt} & \text{if } X_t \geq x_2 \\
A_{gt} + \int_0^U t \frac{F_2^{-1}(dx)}{F_2^{-}[x,\infty)} & \text{if } x_1 < X_t < x_2
\end{cases}
\]
• If the measure $F_{2}^{1-}$ is absolutely continuous with respect to Lebesgue measure with density $f_{2}^{1-}$ then

$$\lambda(t) = \begin{cases} 
0 & \text{if } X_t \geq x_2 \\
\frac{f_{2}^{1-}(U_t)}{F_{2}^{-}[U_t,\infty)} & \text{if } x_1 < X_t < x_2
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is the intensity process (conditional hazard rate), i.e.

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is the intensity process (conditional hazard rate), i.e. $A(t) = \int_0^t \lambda(s)ds$.

• Let $Y_t = E(1_{\{\tau \leq T\}}|\mathcal{F}_t)$. We can find an explicit formula for $Y$ as well.
• We can then calculate prices of risky zero-coupon bonds: $v(t, T)$ is the price at time $t$ of a zero-coupon bond maturing at time $T$. 

\[ v_{\text{mgmt}}(t, T) = E\left[ \{ \delta 1_{\{\tau \leq T\}} \} + (1 - 1_{\{\tau \leq T\}}) \right] e^{-\int_t^T r_s ds} |G_t] = 1 - \left(1 - \delta \right) E[1_{\{\tau \leq T\}} |G_t] e^{-\int_t^T r_s ds} = 1 - \left(1 - \delta \right) p(X_t, t) e^{-\int_t^T r_s ds} \] for $t < T \land \tau$, where the last equality follows from the Markov property.
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Management value of the bond:

$$

\nu^{mgmt}(t, T) = \mathbb{E}\left[\{\delta_1\{\tau \leq T\} + (1 - 1\{\tau \leq T\})\} e^{-\int_t^T r_s ds} | G_t}\right]

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= 1 - [(1 - \delta)\mathbb{E}[1\{\tau \leq T\}| G_t]] e^{-\int_t^T r_s ds}

= 1 - [(1 - \delta)p(X_t, t)] e^{-\int_t^T r_s ds}

for $t < T \wedge \tau$, where the last equality follows from the Markov property.
• Market value of the same bond:

\[ \nu(t, T) = [1-(1-\delta)]E[1\{\tau \leq T\}|F_t]e^{-\int_t^T r_s ds} = [1-(1-\delta)]Y_t e^{-\int_t^T r_s ds} \]
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• In contrast to the management’s using only \(X_t\) and \(T-t\) to determine the price, the market evaluates the price using the following variables: \(X_{gt}\), \(U_t\), \(R(X_t)\), and \(T-t\).
Theoretical considerations for Filtration Shrinkage

- **Question:** If No Free Lunch with Vanishing Risk (NFLVR) holds for \((\Omega, X, \mathcal{G}, P, \mathcal{G})\), does it also hold for \(\mathcal{F}\)?
Theoretical considerations for Filtration Shrinkage

- **Question:** If No Free Lunch with Vanishing Risk (NFLVR) holds for $(\Omega, X, \mathcal{G}, P, \mathcal{G})$, does it also hold for $\mathcal{F}$?

- Recall that NFLVR holds if and only if there exists $Q \sim P$ such that $X$ is a $(Q, \mathcal{G})$ sigma martingale. If $X \geq 0$ a.s., then it is enough that $X$ be a $(Q, \mathcal{G})$ local martingale.
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**Theorem**

*Let \(X\) be a semimartingale for a filtration \(\mathcal{G}\) and let \(\mathcal{F}\) be a subfiltration of \(\mathcal{G}\) such that \(X\) is adapted to \(\mathcal{F}\). Then \(X\) remains a semimartingale for \(\mathcal{F}\).*
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**Theorem**

*Let \(X\) be a positive, \(\mathcal{G}\) local martingale. Let \(F\) be a subfiltration, and assume that \(X\) is adapted to \(F\). Then \(X\) is an \(F\) supermartingale, and if \(X\) is an \(F\) special supermartingale, then \(X\) is an \(F\) local martingale.*
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  Let $X$ be a martingale for a filtration $G$ and let $F$ be any subfiltration of $G$. Then the optional projection of $X$ onto $F$ is again a martingale, for the filtration $F$. 
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  Let $X$ be a local martingale for a filtration $G$ and let $F$ be any subfiltration of $G$. If a sequence of reducing stopping times $(T_n)_{n \geq 1}$ for $X$ in $G$ are also stopping times in $F$, then the optional projection of $X$ onto $F$ is again a local martingale, for the filtration $F$. 
Theorem
If $X$ is a $G$-semimartingale, and $F$ is a subfiltration of $G$, then $\mathcal{O}X$ is an $F$-semimartingale, where $\mathcal{O}X$ denotes the optional projection of $X$ onto $F$.

Theorem
Let $X > 0$ be a $G$-supermartingale. Then $\mathcal{O}X$ is an $F$-supermartingale.

Before stating the next theorem, we need a result of Protter-Shimbo:
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If $X$ is a $\mathcal{G}$ semimartingale, and $\mathcal{F}$ is a subfiltration of $\mathcal{G}$, then $\circ X$ is an $\mathcal{F}$ semimartingale, where $\circ X$ denotes the optional projection of $X$ onto $\mathcal{F}$.

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Theorem

Let $M$ be a locally square integrable martingale such that $\Delta M > -1$. If

$$E \left[ e^{\frac{1}{2} \langle M^c, M^c \rangle_T + \langle M^d, M^d \rangle_T} \right] < \infty,$$  \hspace{1cm} (1)

then $\mathcal{E}(M)$ is martingale on $[0, T]$, where $T$ can be $\infty$. 
Theorem

Let $X > 0$ be a local martingale relative to $(P, \mathbb{G})$. Let $\circ X$ be its optional projection onto a subfiltration $\mathbb{F}$. Then $\circ X$ is a supermartingale, and assume it is special, with canonical decomposition $\circ X_t = 1 + M_t - A_t$. Moreover assume that $\langle M, M \rangle$ exists, and that $dA_t \ll d\langle M, M \rangle_t$. Let $c_s \equiv \frac{dA_s}{d\langle M, M \rangle_s}$ and assume $c_s \Delta M_s > -1$, and

$$
E \left[ e^{1/2 \int_0^T c_s^2 d\langle M^c, M^c \rangle_s + \int_0^T c_s^2 d\langle M^d, M^d \rangle_s} \right] < \infty.
$$

Then there exists a probability $Q$ equivalent to $P$ such that $\circ X$ is a $(Q, \mathbb{F})$ local martingale.
Corollary

Under the hypotheses of the previous theorem, there is NFLVR for $(X, F)$. 