Modelling of successive default events

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The aim of this talk is

- to give a general framework for multi-defaults modelling
- to obtain the dynamics of derivative products in a multiname setting.
The basic tool is the conditional law of the default(s) with respect to a reference filtration
Single Name
Notation

• $G$ is the global market filtration,
• $\tau$ is a default time,
• $H_t = \mathbb{1}_{\tau \leq t}$ is the default processes,
• $H$ is the natural filtration of $H$, with $H \subset G$,
• $F$ is a reference filtration, with $F \subset G$.
On the set $\{\tau > t\}$: ”before-default”

$(\mathcal{F}, \mathcal{G}, \tau)$ satisfy the **minimal assumption (MA)** if

$\forall \, t \geq 0$ and $Z^G \in \mathcal{G}_t$, $\exists \, Z^F \in \mathcal{F}_t$ such that

$$Z^G \cap \{\tau > t\} = Z^F \cap \{\tau > t\}.$$ 

- If $\mathcal{G}^\tau := \mathcal{F} \lor \mathbb{H}$, then $(\mathcal{F}, \mathcal{G}^\tau, \tau)$ enjoys MA.

- Under MA, for any $\mathcal{G}_\infty$-measurable (integrable) r.v. $Y^G$,

$$\mathbb{1}_{\{\tau > t\}} \mathbb{E}[Y^G | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[Y^G \mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t]}{\mathbb{P}(\tau > t | \mathcal{F}_t)} \quad a.s.$$ 

on the set $A := \{\omega : \mathbb{P}(\tau > t | \mathcal{F}_t)(\omega) > 0\}$. 
On the set \( \{ \tau \leq t \} \): ”after-default”

1. We assume that \( G = G^\tau = F \vee H \)

2. **J-Hypothesis** (Jacod for enlargement of filtration purpose):
   
   We assume that there exists a family of \( F_t \otimes B(\mathbb{R}^+) \) r.v.s \( \alpha_t(\theta) \) such that
   
   \[
   \mathbb{P}(\tau \in d\theta|F_t) = \alpha_t(\theta)d\theta \quad d\theta \otimes d\mathbb{P} - a.s.
   \]

   and for any \( \theta \) the process \( \alpha_t(\theta), t \geq 0 \) is a right-continuous martingale

Under the above hypotheses, we can compute \( G_t \)-conditional expectations on the set \( \{ \tau \leq t \} \)

A ”weak version” of J-hypothesis consists of the existence of the density only for \( 0 \leq t \leq \theta \). This is useful for before-default studies, but not sufficient for the after-default ones.
Density and $\mathbb{F}$-martingales

We assume J-hypothesis and introduce the càdlàg (super) martingales

- **Conditional survival process** ($S_t = \mathbb{P}(\tau > t|\mathcal{F}_t), t \geq 0$) (Azéma super-martingale)

- **Conditional probability process** ($S_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t), t \geq 0$).
  Note that, for $t \leq \theta$, one has $S_t(\theta) = \mathbb{E}[S_\theta|\mathcal{F}_t],\$ 

- **Family of martingales** (for $\theta \in \mathbb{R}$): the **densities** ($\alpha_t(\theta), t \geq 0$) of $S_t(\theta)$
Decompositions of Survival process $S$ (super-martingale)

- The **Doob-Meyer decomposition** of $S$ is $S_t = M^F_t - A^F_t$ with
  - the $\mathcal{F}$-martingale
    \[ M^F_t = - \int_0^t (\alpha_t(u) - \alpha_u(u))du = S_t - S_0 + \int_0^t \alpha_u(u)du \]
  - the increasing process $A^F_t = \int_0^t \alpha_u(u)du$

- The **multiplicative decomposition** is
  \[ S_t = L^F_t D^F_t \]
  where
  - The $\mathcal{F}$-martingale $L^F$ is given as $dL^F_t = e^{\int_0^t \lambda^F_s ds} dM^F_t$
  - The decreasing process $D^F$ is $D^F_t = \exp \left( - \int_0^t \lambda^F_s ds \right)$ where $\lambda^F_t = \frac{\alpha_t(t)}{S_t}$ on $\{S_t > 0\}$. 
Links with the classical intensity approach

• The $\mathbb{G}$-intensity is the $\mathbb{G}$-adapted process $\lambda^G$ such that 
  $\left(1_{\{\tau \leq t\}} - \int_0^t \lambda^G_s ds, t \geq 0\right)$ is a $\mathbb{G}$-martingale.

• Under the weak version of J-Hypothesis,
  $\lambda_t^G = 1_{\{\tau > t\}} \lambda_t^F = 1_{\{\tau > t\}} \frac{\alpha_t(t)}{S_t^-} \text{ a.s..}$

• For any $\theta \geq t$,
  $\alpha_t(\theta) = \mathbb{E}[\lambda_{\theta}^G | \mathcal{F}_t] \text{ a.s..}$

Note that the intensity approach does not contain enough information 
to study the after-default case (i.e. for $\theta < t$).
A characterization of $\mathcal{G}$-martingales

Any $\mathcal{G}_t$-measurable r.v. $X$ can be written as

$$X = X_t \mathbb{1}_{\{\tau>t\}} + X_t(\tau) \mathbb{1}_{\{\tau\leq t\}}$$

where $X_t$ and $X_t(\theta)$ are $\mathcal{F}_t$-measurable.

The process $M^X$ is a $\mathcal{G}$-martingale if its decomposition as

$$M^X_t := X_t \mathbb{1}_{\{\tau>t\}} + X_t(\tau) \mathbb{1}_{\{\tau\leq t\}}$$

satisfies

- $(X_t S_t + \int_0^t X_s(s) \alpha_s(s) \, ds, t \geq 0)$ is an $\mathcal{F}$-martingale
- For any $\theta$, $(X_t(\theta) \alpha_t(\theta), t \geq \theta)$ is an $\mathcal{F}$-martingale

Remark: The first condition is equivalent to:

$$X_t L^F_t - \int_0^t (X_s - X_s(s)) L^F_s \lambda^F_s \, ds$$

is an $\mathcal{F}$-martingale
Immersion hypothesis

**Immersion holds if (and only if) any \( \mathbb{F} \)-martingale is a \( \mathbb{G} \)-martingale.**

Under immersion hypothesis,

- \( \alpha_t(\theta) = \alpha_{t \wedge \theta}(\theta) \)
- \( S \) is a non-increasing process
- \( L^\mathbb{F} \) is a constant
- the process
  
  \[
  M_t^X := X_t \mathbb{1}_{\{\tau > t\}} + X_t(\tau) \mathbb{1}_{\{\tau \leq t\}}
  \]

  is a \( \mathbb{G} \)-martingale if

  (a) \( X_t(\theta) \) is a \( \mathbb{F} \)-martingale on \([\theta, \infty)\).

  (b) \( X_t - \int_0^t (X_s - X_s(s)) \lambda_s^\mathbb{F} \, ds \) is an \( \mathbb{F} \)-martingale
Toy Exemple: Cox process model

Let \( \tau = \inf \{ t : \Lambda_t := \int_0^t \lambda_s ds \geq \Theta \} \) where 
\( \Lambda \) is an \( \mathbb{F} \)-adapted increasing process, \( \Lambda_0 = 0, \lim_{t \to \infty} \Lambda_t = +\infty \)

\( \Theta \) is a \( \mathcal{G} \)-measurable r.v. independent of \( \mathcal{F}_\infty \), \( \Theta_i \sim \exp(1) \).

\( \mathcal{F} \) is immersed in \( \mathcal{G} = \mathcal{F} \lor \mathcal{H} \).

The conditional distribution of \( \tau \) is

\[
\begin{cases}
\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{E}[e^{-\Lambda_\theta} | \mathcal{F}_t], & \text{for } \theta > t \\
\mathbb{P}(\tau > \theta | \mathcal{F}_t) = e^{-\Lambda_\theta}, & \text{for } \theta \leq t
\end{cases}
\]

and the density is

\[
\begin{cases}
\alpha_t(\theta) = \mathbb{E}[\lambda_\theta e^{-\Lambda_\theta} | \mathcal{F}_t], & \text{for } \theta > t \\
\alpha_t(\theta) = \lambda_\theta e^{-\Lambda_\theta}, & \text{for } \theta \leq t
\end{cases}
\]
Modelling the density process

Two possible solutions:

- Model $S_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t)$ and then take derivatives w.r.t. $\theta$
- Model the density $\alpha_t(\theta)$ as a family of strictly positive martingales such that $\int_0^\infty \alpha_s(\theta) d\theta = 1$

Remarks:

- for fixed $\theta$, both processes are positive $\mathbb{F}$ martingales
- reference to the interest models
- distinction between $\theta \geq t$ (classical part) and $\theta < t$ (non-classical part)
**F-martingale representation and HJM framework**

Model the family of $\mathbb{F}$-martingales $S_t(\theta)$ in the HJM framework

Suppose

$$
\frac{dS_t(\theta)}{S_t(\theta)} = \Phi_t(\theta)dM_t, \quad t, \theta \geq 0
$$

where $M$ is a continuous multi-dimensional $\mathbb{F}$-martingale, then

$S_t(\theta) = S_t(t) \exp\left(-\int_t^\theta \lambda_t(u)du\right)$ where $\lambda_t(\theta)$ is the forward intensity and

* $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Phi_s(\theta)dM_s - \frac{1}{2} \int_0^t |\Phi_s(\theta)|^2 d\langle M \rangle_s\right)$;

* $S_t = \exp\left(-\int_0^t \lambda^F_s ds + \int_0^t \Phi_s(s)dM_s - \frac{1}{2} \int_0^t |\Phi_s(s)|^2 d\langle M \rangle_s\right)$.

* $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \varphi_s(\theta)dM_s + \int_0^t \varphi_s(\theta)\Phi_s(\theta)d\langle M \rangle_s$. 
For any $\theta \geq 0$, assume that $\lambda_0(\theta)$ is a family of positive probability densities.

Let $b(\theta)$ be a given family of non-negative $\mathbb{F}$-adapted processes. Define

$$\varphi_t(\theta) = -b_t(\theta)\lambda_0(\theta) \exp \left( \int_0^t b_s(\theta) dW_s - \frac{1}{2} \int_0^t b_s(\theta)^2 ds \right)$$

and let

$$\alpha_t(\theta) = \lambda_t(\theta) \exp \left( - \int_0^t \lambda_t(v) dv \right)$$

where $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^\theta \varphi_s(\theta) dW_s + \int_0^t \varphi_s(\theta) \Phi_s(\theta) ds$.

Then the family $(\alpha_t(\theta), t \geq 0)$ is a density process.
Examples of martingale density process

- Compatibility between martingale and probability properties
  - the r.v. $S_t(\theta)$ is $[0, 1]$-valued
  - for any $t$, the map $\theta \to S_t(\theta)$ is non-increasing

- **A Generalized exponential model:** $\forall t, \theta \geq 0$, let

$$S_t(\theta) = \exp \left( -\theta M_t - \frac{1}{2} \theta^2 \langle M \rangle_t \right)$$

where $M$ is an $\mathcal{F}$-martingale.

- Exponential law $S_0(\theta) = \mathbb{P}(\tau > \theta) = \exp(-\theta M_0)$.

- Probability condition: $M_t + \frac{1}{2} \theta \langle M \rangle_t \geq 0$
Comparison with interest rate modelling

- Zero-coupon $B(t, T) = \mathbb{E}[e^{-\int_t^T r_s ds} | \mathcal{F}_t]$.
  Short rate $r_t = -\partial_T |_{T=t} \log B(t, T)$.

- Defaultable zero-coupon without actualization

$$\mathbb{E}[\mathbb{1}_{\tau>T} | \mathcal{G}_t] = \mathbb{1}_{\tau>t} \mathbb{E}[S_T/S_t | \mathcal{F}_t] := \mathbb{1}_{\tau>t} B^\tau(t, T).$$

  Intensity $\lambda_t^\mathbb{F} = -\partial_T |_{T=t} \log B^\tau(t, T)$. 
A general construction of density process. I.

1. Start with $\mathbb{P}_0$ with immersion hypothesis, and $\tau$ with density process $(\alpha_0^0(\theta), t \geq 0)$ constant in time after $\theta$

2. Let $Z_t^G = Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau) \mathbb{1}_{\{\tau \leq t\}}$ a positive $(\mathcal{G}, \mathbb{P}_0)$-martingale with expectation 1

3. Define $d\mathbb{P} = Z_T^G d\mathbb{P}_0$ on $\mathcal{G}_T$. The RN density of $\mathbb{P}$ w.r.t. $\mathbb{P}_0$ on $\mathcal{F}_t$ is $Z_t^F = Z_t S_t + \int_0^t Z_s(\alpha_s^0(s))ds$.

4. Then the **density process of $\tau$ under $\mathbb{P}$** is
   
   $$\alpha_t^P(\theta) = \alpha_0^0(\theta) \frac{Z_t(\theta)}{Z_t^F} \quad \text{for } \theta < t$$
   
   $$\alpha_t^P(\theta) = \mathbb{E}[Z_\theta(\theta)\alpha_\theta^0(\theta)|\mathcal{F}_t] \frac{Z_t^F}{Z_t^F} \quad \text{for } \theta \geq t.$$
A general construction of density process. II.

1. Start with $\mathbb{P}_0$ with $\tau$ independent of $\mathcal{F}_\infty$ with density $f$

2. Let $q_\infty(u)$ a family of $\mathcal{F}_\infty$-measurable r.v. such that
\[ \int_0^\infty q_\infty(u)f(u)du = 1 \]

3. Define $d\mathbb{P} = q_\infty(\tau)d\mathbb{P}_0$ on $\mathcal{G}_\infty$.

4. Then, setting $q_t(u) = \mathbb{E}_0(q_\infty(u)|\mathcal{F}_t)$, the RN density of $\mathbb{P}$ w.r.t. $\mathbb{P}_0$ on $\mathcal{F}_t$ is $Z^\mathbb{F}_t = \int_0^\infty q_t(u)f(u)du$ and the density process of $\tau$ under $\mathbb{P}$ is
\[ \alpha^\mathbb{P}_t(\theta) = q_t(\theta)f(\theta)(Z^\mathbb{F}_t)^{-1} \]
Two ordered default times

Two ordered default times
Notation

Two $G$-stopping times:

$$
\tau = \tau^{(1)} := \min(\tau_1, \tau_2) \quad \text{and} \quad \sigma = \tau^{(2)} := \max(\tau_1, \tau_2).
$$

Before-default and after-default analysis extended naturally to the ordered defaults

- **Filtrations:** $H^{(1)}$ for $\tau$ and $H^{(2)}$ for $\sigma$ respectively. Let
  $$
  G^{(1)} = F \lor H^{(1)} \quad \text{and} \quad G^{(2)} = F \lor H^{(1)} \lor H^{(2)} = G^{(1)} \lor H^{(2)}.
  $$

- On the set $\{ t < \tau \}$, it suffices to apply directly the previous studies
- On the sets $\{ \tau \leq t < \sigma \}$ and $\{ \sigma \leq t \}$ a recursive procedure using $G^{(1)}$ as the reference filtration and $G^{(2)}$ as the global filtration
-conditional survival probability of $\sigma$

- The $\mathbb{G}^{(1)}$-conditional survival probability of $\sigma$ is

$$S_{t}^{\sigma|\mathbb{G}^{(1)}}(\theta) = \mathbb{P}(\sigma > \theta | \mathbb{G}^{(1)}_{t})$$

$$= \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > t, \sigma > \theta | \mathcal{F}_{t})}{\mathbb{P}(\tau > t | \mathcal{F}_{t})} + \mathbb{1}_{\{\tau \leq t\}} \frac{\partial_s \mathbb{P}(\sigma > \theta, \tau > s | \mathcal{F}_{t})}{\partial s \mathbb{P}(\tau > s | \mathcal{F}_{t})} \bigg|_{s = \tau}$$

- We assume that there exists $\alpha_{\tau, \sigma}^{t}$ such that

$$\mathbb{P}(\tau > \theta_1, \sigma > \theta_2 | \mathcal{F}_{t}) = \int_{\theta_1}^{\infty} du_1 \int_{\theta_2}^{\infty} du_2 \alpha_{t, \sigma}^{\tau} (u_1, u_2)$$

- Note that $\alpha_{t, \sigma}^{\tau} (u_1, u_2) = 0$, $\forall u_1 \geq u_2$. 

Two ordered default times
Computation of $\mathbb{G}^{(2)}$-conditional expectations

Explicit formulas on the three sets:

- on $\{t < \tau\}$,

$$
\mathbb{E}[\mathbb{1}_{\{\tau > t\}} Y_T(\tau, \sigma) \mid G_t] = \frac{\mathbb{E}\left[ \int_t^\infty du_1 \int_{u_1}^\infty du_2 Y_T(u_1, u_2) \alpha_T^{\tau,\sigma}(u_1, u_2) \mid F_t \right]}{\int_t^\infty du_1 \int_{u_1}^\infty du_2 \alpha_t^{\tau,\sigma}(u_1, u_2)}
$$

- on $\{\tau \leq t < \sigma\}$,

$$
\mathbb{E}[\mathbb{1}_{\{\tau > t\}} Y_T(\tau, \sigma) \mid G_t] = \frac{\mathbb{E}\left[ \int_t^\infty du_2 Y_T(u_1, u_2) \alpha_T^{\tau,\sigma}(u_1, u_2) \mid F_t \right]}{\int_t^\infty du_2 \alpha_t^{\tau,\sigma}(u_1, u_2)} \bigg|_{u_1=\tau}
$$

- on $\{\sigma \leq t\}$,

$$
\mathbb{E}[\mathbb{1}_{\{\tau > t\}} Y_T(\tau, \sigma) \mid G_t] = \frac{\mathbb{E}[Y_T(u_1, u_2) \alpha_T^{\tau,\sigma}(u_1, u_2) \mid F_t]}{\alpha_t^{\tau,\sigma}(u_1, u_2)} \bigg|_{u_1=\tau \atop u_2=\sigma}
$$
A standard example (Schönbucher-Schubert)

- Cox process model for $\tau_1$ and $\tau_2$:

$$\tau_i = \inf\{t : \Lambda^i_t \geq \Theta_i\}$$

$C$ is the survival copula of $\Theta_1, \Theta_2$

- Marginal survival process $S^i_t = \mathbb{P}(\tau_i > t | \mathcal{F}_\infty) = e^{-\Lambda^i_t}$, immersion hypothesis satisfied for $\mathbb{F}$ and $\mathbb{G}^i$.

- The joint survival probability is obtained from

$$\mathbb{P}(\tau_1 > \theta_1, \tau_2 > \theta_2 | \mathcal{F}_\infty) = C(S^1_{\theta_1}, S^2_{\theta_2}).$$

Therefore

$$S^{1,2}_t(\theta_1, \theta_2) = \mathbb{E}[C(S^1_{\theta_1}, S^2_{\theta_2}) | \mathcal{F}_t].$$
A copula diffusion example

• Joint survival probability

\[ S_0(\theta_1, \theta_2) = \mathbb{P}(\tau_1 > \theta_1, \tau_2 > \theta_2) = \exp \left( - (\theta_1^2 + \theta_2^2)^{\frac{1}{2}} \right) \]

a survival c.d.f of two exponential r.v. with unit parameter linked by a Clayton copula.

• Diffuse the copula function as a martingale: \( \forall t, \theta_1, \theta_2 \geq 0 \), let

\[ S_t(\theta_1, \theta_2) = \exp \left( - \left( \theta_1^2 M_t^1 + \theta_2^2 M_t^2 \right)^{\frac{1}{2}} - A_t \right) \]

where

\[ A_t = \frac{1}{8} \int_0^t \frac{1 + X_s^{\frac{1}{2}}}{X_s^{\frac{3}{2}}} d\langle X \rangle_s \quad \text{and} \quad X_s = \theta_1^2 M_s^1 + \theta_2^2 M_s^2 \]

where \( M^1, M^2 \) positive \( \mathbb{F} \)-martingales s.t. \( \langle M^1, M^2 \rangle_t > 0 \).
Exponential diffusion model

• A two-dimensional exponential example:

\[
\exp \left( -\theta_1 M^1_t - \theta_2 M^2_t - \frac{1}{2} \theta_1^2 \langle M^1 \rangle_t - \frac{1}{2} \theta_2^2 \langle M^2 \rangle_t - \theta_1 \theta_2 \left( \langle M^1, M^2 \rangle_t + a \right) \right)
\]

\( M^1, M^2 \) positive \( \mathbb{F} \) martingales s.t. \( \langle M^1, M^2 \rangle_t \geq 0 \)

• At \( t = 0 \), \( S_0(\theta_1, \theta_2) = \exp(-\theta_1 M^1_0 - \theta_2 M^2_0 - a \theta_1 \theta_2) \).

• Dependence at \( t > 0 \) characterized by \( \langle M^1, M^2 \rangle_t \)

• Probability condition

\[
M^1_t M^2_t - \langle M^1, M^2 \rangle_t > a > 0
\]
Several Defaults, Applications to pricing

• Generalization to $n$ successive defaults $\sigma_1 \leq \cdots \leq \sigma_n$ by a recursive method

• Representation of conditional expectation with respect to $G_t^{(1, \cdots, n)} = F_t \lor H_{t1} \lor \cdots \lor H_{tn}$

Let $Y_t(u_1, \cdots, u_n)$ be a family of r.v. $F_t \otimes B(\mathbb{R}^n)$-measurable where $t, u_1, \cdots, u_n \geq 0$. Then

$$\mathbb{E}[Y_T(\sigma_1, \cdots, \sigma_n) | G_t^{(1, \cdots, n)}] = \sum_{i=0}^{n} \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} q_t^i(T, \sigma_1, \cdots, \sigma_i, Y_T)$$

where $q_t^i(T, s_1, \cdots, s_i, Y_T)$ is a ratio of $F_t$ conditional expectations, $\sigma_0 = 0$ and $\sigma_{n+1} = \infty$. 
Several Defaults, Applications to pricing

Pricing of the portfolio credit derivatives

- \( k^{th} \)-to-default swap depends on the \( k^{th} \) default time of the underlying portfolio:

\[
\mathbb{E}\left[ \mathbb{1}_{\{\sigma_k > T\}} Y_T | \mathcal{G}_t^{(1,\ldots,n)} \right] = \sum_{i=0}^{k-1} \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} q_{t,Q}^i(T, \sigma_1, \ldots, \sigma_i, Y_T, S_T^{(k)})
\]

where \( S_T^{(k)} = \mathbb{P}(\sigma_k > T | \mathcal{F}_t) \).

- For a CDO tranche, total loss \( l_T = \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \leq t\}} \) and key term to calculate:

\[
\mathbb{E}\left[ (K - l_T)^+ | \mathcal{G}_t^{(1,\ldots,n)} \right] = \int_{-\infty}^{K} du \mathbb{E}\left[ \mathbb{1}_{\{\sigma_{\lfloor u \rfloor + 1} > T\}} | \mathcal{G}_t^{(1,\ldots,n)} \right]
\]

\[
= \int_{-\infty}^{K} du \sum_{i=0}^{\lfloor u \rfloor} \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} q_{t,Q}^i\left(T, \sigma_1, \ldots, \sigma_i, S_T^{\lfloor u \rfloor}\right).
\]
Dynamics of CDSs prices

Let $X(\kappa)$ be the price of a CDS written on the default $\tau_1$, with recovery $\delta$ and premium $\kappa$ and $X_t(\kappa) = \mathbb{1}_{t \leq \tau_2 \wedge \tau_1} \tilde{X}_t(\kappa) + \mathbb{1}_{\tau_2 < t \leq \tau_1} \hat{X}_t(\kappa)$ its price. Let

$$S_t(u, v) = \mathbb{P}(\tau_1 > u, \tau_2 > v | F_t) = S_0(u, v) + \int_0^t g_s(u, v) dW_s$$

Then:

$$d\tilde{X}_t(\kappa) = \frac{1}{S_t(t, t)} \left[ \left( \delta(t) \partial_1 S_t(t, t) + \kappa S_t(t, t) - (\partial_1 S_t(t, t) + \partial_2 S_t(t, t)) \tilde{X}_t(\kappa) \right) dt ight.$$

$$- \left( \int_t^T (\delta(u) \alpha_t(u, t) + \kappa \partial_2 S_t(u, t)) du \right) dt + \sigma_t(T) d\hat{W}_t \left. \right]$$

with

$$\sigma_t(T) = -\frac{1}{S_t(t, t)} \left( \int_t^T (\delta(u) \partial_1 g_t(u, t) + \kappa_1 g_t(u, t)) du + g_t(t, t) \tilde{X}_t(\kappa) \right).$$
\[ d\hat{X}_t(\kappa) = \left( -\delta(t)\lambda^{1\mid 2}(t, \tau_2) + \kappa + \hat{X}_t(\kappa)\lambda^{1\mid 2}(t, \tau_2) \right) dt + \sigma^{1\mid 2}_t(T) d\hat{W}_t \]

where

\[
\lambda^{1\mid 2}(t, s) = -\frac{\alpha_t(t, s)}{\partial_2 S_t(t, s)} \\
\sigma^{1\mid 2}_t(T) = \frac{1}{\partial_2 S_t(t, \tau_2)} (A_t(\tau_2) - \hat{X}_t(\kappa)\partial_2 g_t(t, \tau_2)), \\
A_t(s) = -\int_t^T \delta(u)\partial_{12} g_t(u, s) du - \kappa \int_t^T \partial_2 g_t(u, s) du.
\]
Perspectives

A general framework for portfolio of defaultable names:

- explicit model studies for the joint density process
- application to the pricing
- calibration of parameters
- dynamic hedging