Uniform Continuity is Almost Lipschitz Continuity *

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January 15, 1997

Abstract

We prove that uniformly continuous functions on convex sets are almost Lipschitz continuous in the sense that \( f \) is uniformly continuous if and only if, for every \( \epsilon > 0 \), there exists a \( K < \infty \), such that \( \|f(y) - f(x)\| \leq K\|y - x\| + \epsilon \).

*AMS 1980 Subject Classifications: Primary 54E15; Secondary 54C05, 46B99, 26B05
In this paper, we are interested in the relationship between two important subclasses of continuous functions: uniformly-continuous functions and Lipschitz-continuous functions. Throughout, we assume that all functions map a domain $D$ of one normed linear space into another.

Recall the definition of uniform continuity:

**Definition 1** A function $f$ is uniformly continuous if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that $\|f(y) - f(x)\| < \epsilon$ whenever $\|y - x\| < \delta$. 
The definition of Lipschitz continuity is also familiar:

**Definition 2** A function $f$ is Lipschitz continuous if there exists a $K < \infty$ such that $\|f(y) - f(x)\| \leq K\|y - x\|$.

It is easy to see (and well-known) that Lipschitz continuity is a stronger notion of continuity than uniform continuity. For example, the function $f(x) = x^{1/3}$ on $\mathbb{R}$ is uniformly continuous but not Lipschitz continuous. Hence, it is perhaps surprising to note that uniformly continuous functions are almost Lipschitz:

**Theorem 1** A function $f$ defined on a convex domain is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a $K < \infty$ such that $\|f(y) - f(x)\| \leq K\|y - x\| + \epsilon$.

**Proof.** Suppose that $f$ is uniformly continuous on a convex domain $D$ and fix $\epsilon > 0$. Then, there exists a $\delta > 0$ such that $\|f(z) - f(z')\| < \epsilon$ whenever $\|z - z'\| < \delta$. Fix $x$ and $y$ in $D$ and let

$$z_k = x + k\frac{\delta}{2} \frac{y - x}{\|y - x\|} \quad \text{for } k = 0, 1, 2, \ldots, N$$

where

$$N = \left\lfloor \frac{\|y - x\|}{\delta/2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. From the convexity of $D$, we see that each $z_k$ belongs to $D$. Also,

$$z_0 = x,$$

$$\|z_k - z_{k-1}\| = \delta/2,$$

and

$$\|y - z_N\| < \delta/2.$$
Hence,

\[ \| f(y) - f(x) \| \leq \sum_{k=1}^{N} \| f(z_k) - f(z_{k-1}) \| + \| f(y) - f(z_N) \| \]
\[ < (N + 1)\epsilon \]
\[ \leq \frac{2\epsilon}{\delta} \| y - x \| + \epsilon. \]

Picking \( K = 2\epsilon/\delta \) establishes the “only if” direction.

For the “if” part, suppose that \( f \) satisfies the condition given in the theorem. Fix \( \epsilon > 0 \) and choose \( K \) so that

\[ \| f(y) - f(x) \| \leq K\| y - x \| + \epsilon/2 \]

for all \( x, y \in D \). Put \( \delta = \epsilon/2K \). If \( \| y - x \| < \delta \), then we see that

\[ \| f(y) - f(x) \| < \epsilon. \]

Hence, \( f \) is uniformly continuous. \( \square \)

The author originally discovered this result when puzzling over why a uniformly continuous function can have unbounded slopes (such as in the cube root example), but cannot grow faster than linearly. The \( \epsilon \) in our “almost Lipschitz” characterization allows for unbounded slopes locally, but nevertheless, it is now easy to see that the growth at infinity is at most linear. Indeed, suppose that 0 is a point in \( D \), and apply our characterization to 0 and \( y \) to see that

\[ \| f(y) \| \leq \| f(0) \| + \| f(y) - f(0) \| \]
\[ \leq \| f(0) \| + K\| y \| + \epsilon \]

which shows that \( \| f(y) \| \leq a + b\| y \| \) for an appropriate choice of \( a \) and \( b \).

To illustrate the use of this new characterization of uniform continuity, we reprove one of the standard theorems (see e.g. [Roy88]) about uniformly continuous functions:
Theorem 2 If $\{x_n : n = 0, 1, 2, \ldots\}$ is a Cauchy sequence and $f$ is uniformly continuous (on a convex domain $D$), then $\{f(x_n) : n = 0, 1, 2, \ldots\}$ is also a Cauchy sequence.

Proof. Fix an arbitrary $\epsilon > 0$. Since $f$ is uniformly continuous, there exists a constant $K < \infty$ such that

$$\|f(x_n) - f(x_m)\| \leq K\|x_n - x_m\| + \epsilon.$$ 

Since $\{x_n\}$ is a Cauchy sequence, $\lim_{N \to \infty} \sup_{n,m \geq N} \|x_n - x_m\| = 0$. Hence,

$$0 \leq \lim_{N \to \infty} \sup_{n,m \geq N} \|f(x_n) - f(x_m)\| \leq \epsilon.$$ 

But $\epsilon$ was arbitrary and so the lim sup actually vanishes which proves that $\{f(x_n)\}$ is a Cauchy sequence. \qed

Acknowledgement. The author would like to thank Sid Browne for encouraging him to write this short note.

References