Linear Stability of Ring Systems

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2007 March 5

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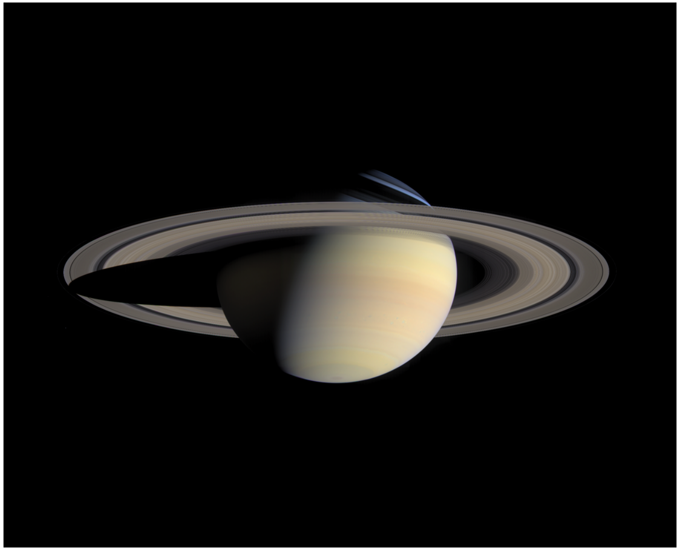
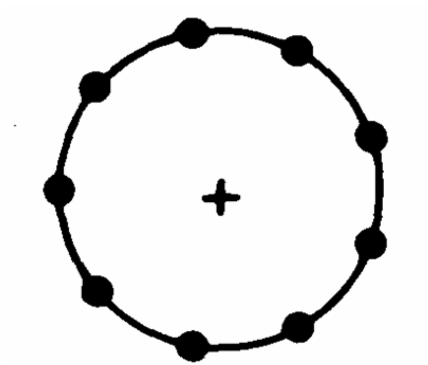
Saturn's Rings



Beautiful Saturn



Simplified model of a ring system

In 1859, J.C. Maxwell won the prestigious Adams Prize.

His Results:

- Rings of Saturn must be composed of small particles.
- Modeled the ring as n co-orbital particles of mass m.
- For large n, ring system is stable if

$$\frac{m}{M} \leq \frac{2.298}{n^3}$$

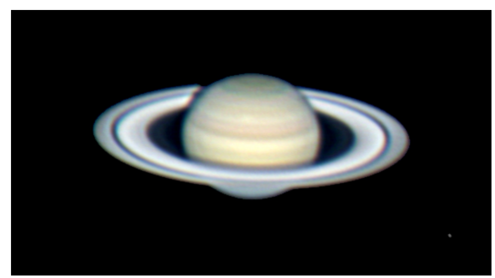
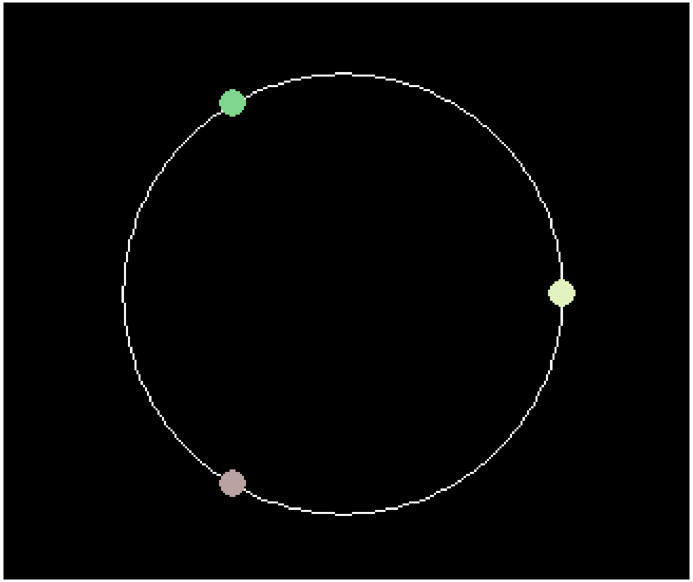
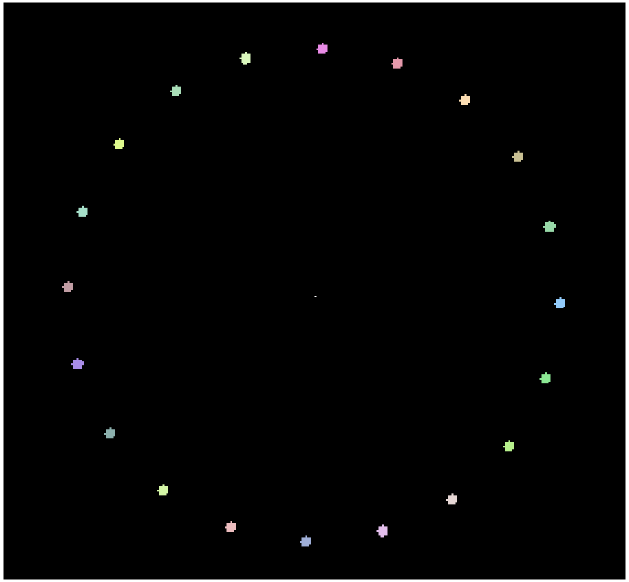


Image From Earth

Ring Systems Alone Are Unstable





Theorem 1 *The system is stable if and only if* $n = 2$.

NOTE: On this and subsequent pages, most graphic images are links to JAVA *applets that animate the motion. Click on 'em.*

A Large Central Mass Stabilizes

Equation of motion for $j = 0, \ldots, n-1$

$$\ddot{z}_j = GM\frac{z_n - z_j}{|z_n - z_j|^3} + \sum_{k \neq j,n} Gm\frac{z_k - z_j}{|z_k - z_j|^3}.$$

About center of mass

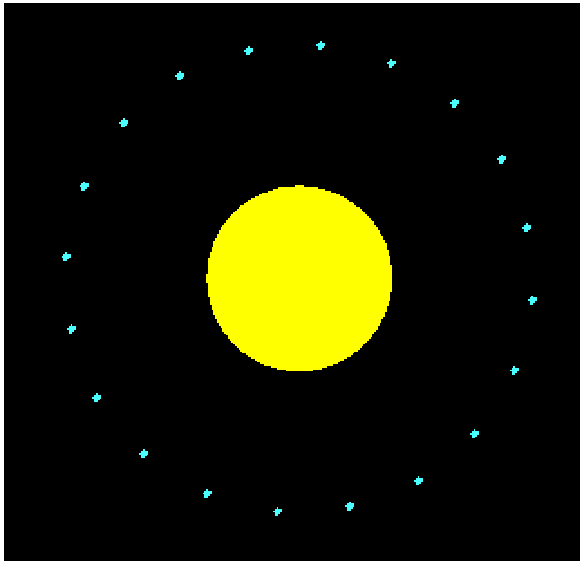
$$z_n = -\frac{m}{M}\sum_{j=0}^{n-1} z_j.$$

Equilibrium point

$$\begin{aligned} z_j(t) &= re^{i(\omega t + 2\pi j/n)}, \qquad j = 0, \ldots, n-1 \\ z_n(t) &= 0, \end{aligned}$$

where

$$\omega^2 = \frac{GM}{r^3} + \frac{Gm}{4r^3}\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)}.$$



20 Janus mass moons

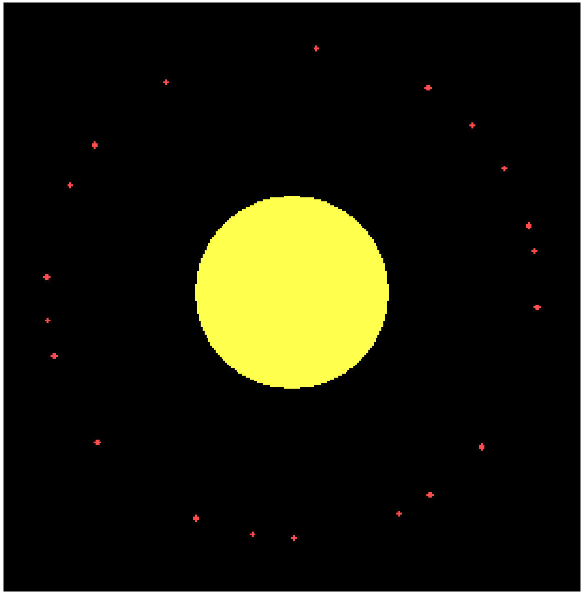
Stable! WHY?

Slight Perturbation

Here's 20 Janus masses

Orbits are initialized to be circular

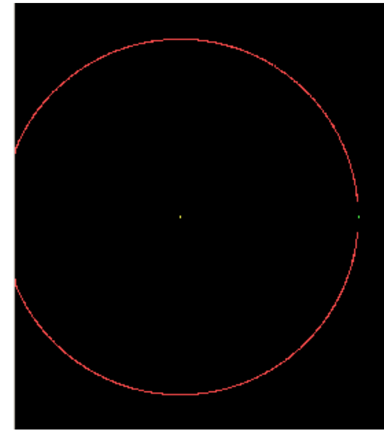
Distances from Saturn are randomized (only slightly)

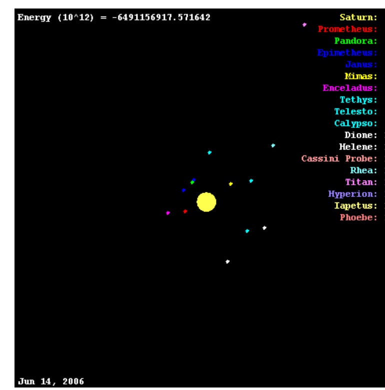


Consider Moons Janus and Epimetheus

They "Horseshoe"

- Orbital radii that differ by 55km.
- Orbit Saturn once every 17 hours.
- Lower one orbits slightly faster.
- Every four years, the lower one begins to overtake the higher one.
- As it slowly creeps up from behind, their mutual gravitational pull speeds up the lower one and slows down the higher.
- The higher one then drops to a lower orbit and speeds up.
- The lower swings up into a higher orbit and slows down.
- Their roles have switched—fast one does not pass slow one.
- The process continues indefinitely.





Main Results

Theorem 2

- *For $2 \leq n \leq 6$, the ring system is unstable.*
- *For $n \geq 7$, the ring system is (linearly) stable if and only if*

$$\frac{m}{M} \leq \frac{\gamma_n}{n^3}.$$

- $\lim_{n\to\infty} \gamma_n = 2.2987$.

Simulation confirms the stability analysis:

n	γ_n	Simulator
2	*	[0.0, 0.007]
6	*	[0.0, 0.025]
7	2.452	[2.45, 2.46]
8	2.4121	[2.41, 2.42]
10	2.3753	[2.37, 2.38]
12	2.3543	[2.35, 2.36]
14	2.3411	[2.34, 2.35]
20	2.3213	[2.32, 2.33]
36	2.3066	[2.30, 2.31]
50	2.3031	[2.30, 2.31]
100	2.2999	[2.30, 2.31]
500	2.2987	

If You Really Have to Know...

The Formula for γ_n is explicit but ugly

$$
\begin{aligned}
n^3/\gamma_n \;=\; & 2(J_n - \tilde{J}_{n/2\pm 1,n}) + \frac{9}{2}(J_n - \tilde{J}_{n/2,n}) - 5I_n \\
& + \sqrt{\left(2(J_n - \tilde{J}_{n/2\pm 1,n}) + \frac{9}{2}(J_n - \tilde{J}_{n/2,n}) - 4I_n\right)^2 - \frac{9}{4}\left(J_n - \tilde{J}_{n/2,n}\right)^2},
\end{aligned}
$$

where

$$
I_n \;=\; \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\sin\left(\pi k/n\right)} \approx \frac{n}{2\pi}\sum_{k=1}^{(n-1)/2}\frac{1}{k} \approx \frac{n}{2\pi}\log(n/2)
$$

$$
J_n \;=\; \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\sin^3\left(\pi k/n\right)} \approx \frac{n^3}{2\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{n^3}{2\pi^3}\zeta(3) = 0.01938\, n^3
$$

$$
\tilde{J}_{j,n} \;=\; \frac{1}{4}\sum_{k=1}^{n-1}\frac{\cos(2\pi kj/n)}{\sin^3(\pi k/n)} \quad\Longrightarrow\quad \tilde{J}_{n/2,n} \approx -\frac{3}{4}J_n
$$

Linear Stability Analysis

Counter rotate (and map to positive real axis):

$$w_j = e^{-i(\omega t + 2\pi j/n)} z_j.$$

Treating w_j and $\bar{w}_j$ as independent variables, put

$$W_j = \begin{bmatrix} w_j \\ \bar{w}_j \end{bmatrix}.$$

Linearize equation of motion around equilibrium point:

$$\frac{d}{dt}\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} \approx \left[\begin{array}{cccc|cccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \\ \hline D & N_1 & \cdots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ N_1 & N_2 & \cdots & D & & & & \Omega \end{array}\right] \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}$$

Stability is Determined by Eigenvalues of $4n \times 4n$ System

$$\left[\begin{array}{cccc|cccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \\ \hline D & N_1 & \cdots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ N_1 & N_2 & \cdots & D & & & & \Omega \end{array}\right] \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} = \lambda \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}.$$

First $2n$ equations give

$$\delta \dot{W}_j = \lambda \delta W_j$$

Substituting, we get a *block circulant matrix*:

$$\begin{bmatrix} D & N_1 & \cdots & N_{n-1} \\ N_{n-1} & D & \cdots & N_{n-2} \\ \vdots & \vdots & & \vdots \\ N_1 & N_2 & \cdots & D \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} + \lambda \begin{bmatrix} \Omega & & & \\ & \Omega & & \\ & & \ddots & \\ & & & \Omega \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \lambda^2 \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix}.$$

Block Circulant Matrix

Look for solutions of the form:

$$\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \begin{bmatrix} \xi \\ \rho_j \xi \\ \vdots \\ \rho_j^{n-1} \xi \end{bmatrix},$$

where ρ_j is an n-th root of unity

$$\rho_j = e^{2\pi i j/n}.$$

The $2n \times 2n$ system then reduces to n 2×2 systems the determinant of which must vanish:

$$\det\left(D + \sum_{k=1}^{n-1} \rho_j^k N_k + \lambda \Omega - \lambda^2 I \right) = 0.$$

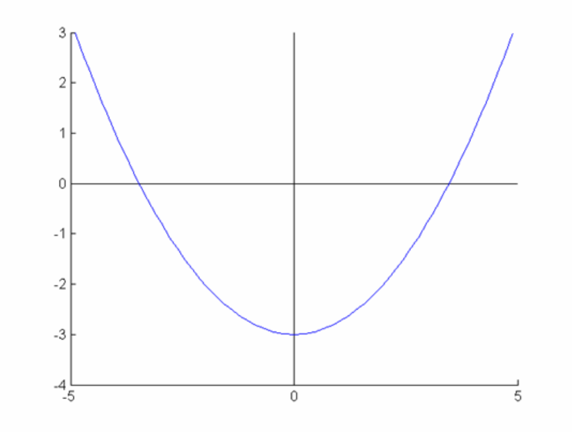
Replacing λ with $i\lambda$, we get a characteristic polynomial with real coefficients

$$f(\lambda) = \lambda^4 + A_j \lambda^2 + B_j \lambda + C_j = 0.$$

Find when this equation has 4 real roots.

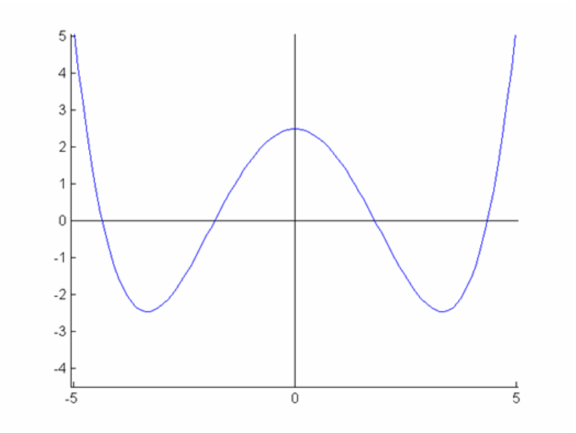
Counting Real Roots of $f(\lambda) = \lambda^4 + A_j\lambda^2 + B_j\lambda + C_j = 0$

For $2 \leq n \leq 6$ and $j = 1$, $f(\lambda)$ has this form:



Hence, there can be at most 2 real roots and so the system is always unstable.

For $n \geq 7$ and all j, $f(\lambda)$ has this form:



Hence, there can be 4 real roots and so we have the *possibility of stability*.

If $j = n/2$ has four real roots, then so do all other polynomials.

Details are tedious, but analysis of the $j = n/2$ case produces the threshold γ_n given earlier.

Density Estimate

Let

$$\lambda \;=\; \text{linear density of the masses} = \frac{\text{diam of a boulder}}{\text{separation between boulders}}$$

If δ denotes the boulders' density, then the mass of a boulder is

$$m = (4\pi/3)(\lambda \pi r/n)^3 \delta.$$

The density of the boulders in Saturn's rings is about $1/8$ of Earth's density

$$\delta = \frac{1}{8} \frac{M_E}{(4\pi/3) r_E^3}.$$

Recall our stability threshold

$$m \leq 2.298 M/n^3.$$

Combining, we get an inequality *without* n:

$$\left(\lambda \pi \frac{r}{r_E} \right)^3 \leq (8)(2.298) \left(\frac{M_S}{M_E} \right)$$

Substituting $r = 120,000$km and $M_S = 95.5 M_E$ and solving for λ, we get

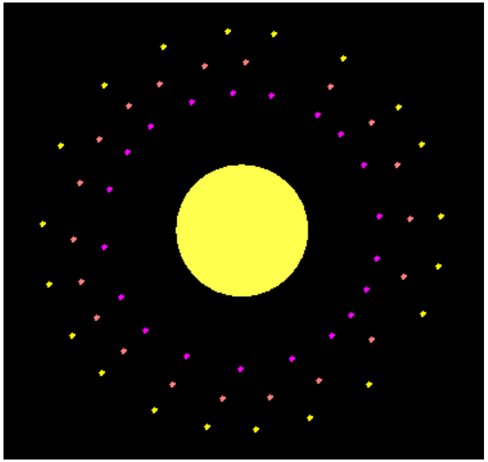
$$\lambda \leq 20.4\%.$$

Remark: Gravity scales correctly—a marble orbits a bowling ball every 90 minutes.

Rings at Multiple Radii

General principle: it is easier for a body to destabilize bodies at the same radius from the central mass.

Hence, if each of many single rings are stable, then one might expect the entire system to be stable.



Mathematical verification is profoundly difficult—no longer does a single counter-rotation freeze all bodies.

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