Linear Stability of Ring Systems Around Oblate Central Masses

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ABSTRACT

In this paper, we consider the stability of a ring of bodies of equal mass uniformly distributed around a large oblate central mass. The purpose of this and previous papers is to shed light on the stability of Saturn's rings. Previous papers have been limited by the assumptions that (1) all ring bodies are at the same distance from the central body, (2) the central body acts like a point mass (i.e., is a perfect sphere), and (3) the ring bodies all have the same mass and are evenly spaced around the ring. The third limitation is probably the least important; as long as there are a large number of masses and the mass distribution is approximately uniform then the system should behave as a system of equally-spaced, equal-mass bodies. The first main purpose of this paper is to remove the second limitation. But, the paper also aims to address limitation (1). Recent computer simulations of single-ring systems have shown that the threshold for stability, as determined by a linear stability analysis, matches precisely the stability threshold predicted by simulation. In other words, a linear stability analysis while presumably just a mathematical abstraction actually tells us something quite real. Furthermore, simulations of multi-ring systems suggest that instability comes from azimuthal perturbations; small azimuthal changes are more destabilizing that small radial perturbations. Hence, in this paper, we also consider the situation where the central body consists not just of an oblate central mass but also incorporates a flat ring representing in aggregate all ring bodies at radii other than the one under consideration. The central oblate body together with a flat ring is modeled simply by introducing two oblateness terms to the gravitational potential associated with the central mass. The subsequent analysis is almost identical to the case of a single oblateness term. For Saturn, the oblateness of the central mass is six orders of magnitude more significant than the rings at other radii as a destabilizing influence.

Subject headings: planets: rings

1. Introduction

Saturn's rings have proved to be both mysterious and inspirational from the time they were discovered by Galileo Galilei in 1610 to the present day with the Cassini probe sending back to Earth a steady stream of beautiful new images. Maxwell (1859) won the Adams Prize for his efforts to understand the stability of a ring such as Saturn's. Maxwell did not know whether the ring consisted of a myriad of small independent bodies (as it does) or was a one-piece solid ring. He studied both of these cases. For the multiple independent body case, he assumed, for simplicity, a single ring of equal-mass bodies orbiting a massive central body, which was assumed to be acting as a point mass at the center of the system. For this particular case, he assumed that the perturbation most likely to lead to instability was the alternate advancement and retardation to each successive body in its orbit. With this assumption he was able to work out a specific inequality telling when the system could be stable. His inequality is

$$m \leq 2.298M/n^3,$$

where m denotes the mass of one ring body, M denotes the mass of the central body, and n denotes the number of ring bodies. Subsequent work over the intervening 150 years has filled in some gaps in Maxwell's argument. In particular, it is now possible to do a complete eigenvalue decomposition of the linear stability matrix and show that, for large n, Maxwell found exactly the correct answer. Even more remarkable is the fact that the linear stability analysis carried out by Maxwell and subsequent investigators actually does indentify the threshold between stability and instability. Normally, a stability analysis gives only a necessary condition for stability. But, as shown experimentally in Vanderbei and Kolemen (2007), the condition is evidently both necessary and sufficient.

The purpose of this paper is to consider cases where the central mass does not act like a point mass. Specifically, we introduce the oblateness term to the gravitational potential produced by the central mass. Such a modification brings the analysis one step closer to a model of a true ring system such as one finds around Saturn.

In the last section of the paper, we introduce a second oblateness term to the gravitational potential associated with the central mass. As before, the first oblateness term is intended to account for the oblateness of a large and rapidly spinning central body. The second oblateness term represents a flat disk having a much smaller mass but a radius that extends out beyond the orbit of the ring of particles being studied. This second disk-like oblate object represents the net effect of all of the ring particles at radii other than the radius under consideration. We treat this disk as a solid object firmly attached to the central body—see Figure 1. Our interest is not whether such a body would be stable as an independent object orbiting a central mass. Rather, we are interested only in the effect of such a body on the stability of a specific ring of equal-mass objects orbiting at a given radius. Numerical simulations suggest that instability is produced by interactions between pairs of bodies at the same radius; not pairs of bodies at different radii. Hence, it seems entirely reasonable to analyze a system where the effect of all ring bodies orbiting at a different radius are considered at once as a single flat disk of “particles”.

2. Equally-Spaced, Equal-Mass Bodies in a Circular Ring About an Oblate Massive Body

Consider the multibody problem consisting of one large oblate central body, say Saturn, having mass M, oblateness $\mathcal{J}_2$, and equatorial radius R and n small bodies, such as boulders, each of mass m orbiting the large body in circular orbits uniformly spaced in a ring of radius r. Indices 0 to $n-1$ will be used to denote the ring masses and index n will be used for Saturn. Throughout the paper we assume that $n \geq 2$.

The purpose of this section is to show that such a ring exists as a solution to Newton’s law of gravitation. In particular, we derive the relationship between the angular velocity ω of the ring particles and their radius r from the central mass. We assume all bodies lie in a plane and therefore complex-variable notation is convenient. So, with $i = \sqrt{-1}$ and $z = x + iy$, we can write the equilibrium solution for $j = 0, 1, \ldots, n-1$, as

$$z_j = re^{i(\omega t + 2\pi j/n)} \tag{1}$$

and

$$z_n = 0. \tag{2}$$

By symmetry (and exploiting our assumption that $n \geq 2$), force is balanced on Saturn itself. Now consider the ring bodies. Differentiating (1), we see that

$$\ddot{z}_j = -\omega^2 z_j. \tag{3}$$

From Newton’s law of gravity we have that

$$\ddot{z}_j = -GM\frac{z_j - z_n}{|z_j - z_n|^3} - \frac{3}{2}\mathcal{J}_2 R^2 GM \frac{z_j - z_n}{|z_j - z_n|^5} + \sum_{k \neq j,n} Gm \frac{z_k - z_j}{|z_k - z_j|^3}. \tag{4}$$

Equations (3) and (4) allow us to determine ω, which is our first order of business. By symmetry it suffices to consider $j = 0$. It is easy to check that

$$z_k - z_0 = re^{i\omega t}e^{\pi i k/n} 2i \sin(\pi k/n) \tag{5}$$

and hence that

$$|z_k - z_0| = 2r\sin(\pi k/n). \tag{6}$$

Substituting (5) and (6) into (4) and equating this with (3), we see that

$$\begin{aligned} -\omega^2 &= -\frac{GM}{r^3} - \frac{3}{2}\mathcal{J}_2 R^2 \frac{GM}{r^5} + \sum_{k=1}^{n-1} \frac{Gm}{4r^3} \frac{ie^{\pi ik/n}}{\sin^2(\pi k/n)} \\ &= -\frac{GM}{r^3} - \frac{3}{2}\mathcal{J}_2 R^2 \frac{GM}{r^5} - \frac{Gm}{4r^3}\sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)} + i\frac{Gm}{4r^3}\sum_{k=1}^{n-1}\frac{\cos(\pi k/n)}{\sin^2(\pi k/n)}. \end{aligned} \tag{7}$$

It is easy to check that the summation in the imaginary part on the right vanishes. Hence,

$$\begin{aligned} \omega^2 &= \frac{GM}{r^3} + \frac{3}{2}\mathcal{J}_2 R^2 \frac{GM}{r^5} + \frac{Gm}{r^3} I_n \\ &= \frac{GM}{r^3}\left(1 + \frac{3}{2}\mathcal{J}_2\,(R/r)^2\right) + \frac{Gm}{r^3} I_n \end{aligned} \tag{8}$$

where

$$I_n = \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)}. \tag{9}$$

With this choice of ω, the trajectories given by (1) and (2) satisfy Newton's law of gravitation.

3. First-Order Stability

In order to carry out a stability analysis, we need to counter-rotate the system so that all bodies remain at rest. We then perturb the system slightly and analyze the result.

A counter-rotated system would be given by

$$e^{-i\omega t} z_j(t) = re^{2\pi ij/n} = z_j(0).$$

In such a rotating frame of reference, each body remains fixed at its initial point. It turns out to be better to rotate the different bodies different amounts so that every ring body is repositioned to lie on the x-axis. In other words, for $j = 0, \ldots, n-1, n$, we define

$$w_j = u_j + iv_j = e^{-i(\omega t + 2\pi j/n)} z_j. \tag{10}$$

The advantage of repositioning every ring body to the positive real axis is that perturbations in the real part for any ring body represent radial perturbations whereas perturbations in the imaginary part represent azimuthal perturbations. A simple counter-rotation does not provide such a clear distinction between the two types of perturbations (and the associated stability matrix fails to have the circulant property that is crucial to all later analysis).

Differentiating (10) twice, we get

$$\ddot{w}_j = \omega^2 w_j - 2i\omega\dot{w}_j + e^{-i(\omega t+2\pi j/n)}\ddot{z}_j. \tag{11}$$

From Newton's law of gravity, we see that

$$\ddot{w}_j = \omega^2 w_j - 2i\omega\dot{w}_j + \sum_{k\neq j} Gm_k \frac{\xi_{k,j}}{|\xi_{k,j}|^3} + \frac{3}{2}GM\mathcal{J}_2R^2\frac{\xi_{n,j}}{|\xi_{n,j}|^5}, \tag{12}$$

where

$$m_k = \begin{cases} m, & \text{for } k = 0,1,\ldots,n-1, \\ M, & \text{for } k = n, \end{cases} \tag{13}$$

$$\xi_{k,j} = e^{i\theta_{k-j}}w_k - w_j \tag{14}$$

and

$$\theta_k = 2\pi k/n. \tag{15}$$

Let $\delta w_j(t)$ denote variations about the fixed point given by

$$w_j \equiv \begin{cases} r, & \text{for } j = 0,1,\ldots,n-1, \\ 0, & \text{for } j = n. \end{cases} \tag{16}$$

We compute a linear approximation to the differential equation describing the evolution of such a perturbation. Applying the quotient, chain, and product rules as needed, we get

$$\begin{aligned} \delta\ddot{w}_j &= \omega^2\delta w_j - 2i\omega\delta\dot{w}_j + \sum_{k\neq j} Gm_k \frac{|\xi_{k,j}|^3\delta\xi_{k,j} - \xi_{k,j}\frac{3}{2}|\xi_{k,j}|(\xi_{k,j}\delta\bar{\xi}_{k,j} + \bar{\xi}_{k,j}\delta\xi_{k,j})}{|\xi_{k,j}|^6} \\ &\quad + \frac{3}{2}GM\mathcal{J}_2R^2\frac{|\xi_{n,j}|^5\delta\xi_{n,j} - \xi_{n,j}\frac{5}{2}|\xi_{n,j}|^3(\xi_{n,j}\delta\bar{\xi}_{n,j} + \bar{\xi}_{n,j}\delta\xi_{n,j})}{|\xi_{n,j}|^{10}} \\ &= \omega^2\delta w_j - 2i\omega\delta\dot{w}_j - \frac{1}{2}\sum_{k\neq j} Gm_k \frac{|\xi_{k,j}|^2\delta\xi_{k,j} + 3\xi_{k,j}^2\delta\bar{\xi}_{k,j}}{|\xi_{k,j}|^5} \\ &\quad - \frac{1}{2}\,\frac{3}{2}GM\mathcal{J}_2R^2\frac{3|\xi_{n,j}|^2\delta\xi_{n,j} + 5\xi_{n,j}^2\delta\bar{\xi}_{n,j}}{|\xi_{n,j}|^7}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \delta\xi_{k,j} &= e^{i\theta_{k-j}}\delta w_k - \delta w_j \\ \delta\bar{\xi}_{k,j} &= e^{-i\theta_{k-j}}\delta\bar{w}_k - \delta\bar{w}_j. \end{aligned}$$

The next step is to use (14) to re-express the $\xi_{k,j}$'s in terms of the w_k's and the w_j's and then to substitute in the particular solution given by (16). Consider the case where $j < n$. In this case we have

$$\xi_{k,j} = \begin{cases} r(e^{i\theta_{k-j}} - 1), & \text{for } k < n, \\ -r, & \text{for } k = n \end{cases}$$

and therefore

$$|\xi_{k,j}| = \begin{cases} 2r\sin(|\theta_{k-j}|/2), & \text{for } k < n, \\ r, & \text{for } k = n. \end{cases}$$

Substituting these into (17) and simplifying, we get

$$\begin{aligned} \delta\ddot{w}_j &= \omega^2 \delta w_j - 2i\omega\delta\dot{w}_j - \frac{GM}{2r^3}(e^{-i\theta_j}\delta w_n + 3e^{i\theta_j}\delta\bar{w}_n) + \frac{GM}{2r^3}(\delta w_j + 3\delta\bar{w}_j) \\ &\quad - \frac{Gm}{2r^3}\,\frac{1}{8}\sum_{k\neq j,n} \frac{e^{i\theta_{k-j}}\delta w_k - \delta w_j - 3e^{i\theta_{k-j}}(e^{-i\theta_{k-j}}\delta\bar{w}_k - \delta\bar{w}_j)}{\sin^3(|\theta_{k-j}|/2)} \\ &\quad - \frac{3}{4}\,\frac{GM\mathcal{J}_2R^2}{r^5}(3e^{-i\theta_j}\delta w_n + 5e^{i\theta_j}\delta\bar{w}_n) + \frac{3}{4}\,\frac{GM\mathcal{J}_2R^2}{r^5}(3\delta w_j + 5\delta\bar{w}_j). \end{aligned} \qquad (18)$$

4. Choice of Coordinate System

Without loss of generality, we can choose our coordinate system so that the center of mass remains fixed at the origin. Having done that, the perturbations δw_n and $\delta\bar{w}_n$ can be computed explicitly in terms of the other perturbations. Indeed, conservation of momentum implies that

$$m\sum_{k\neq n}\delta z_k + M\delta z_n = 0.$$

Hence,

$$\delta z_n = -\frac{m}{M}\sum_{k\neq n}\delta z_k.$$

From the definition (10) of the w_k's in terms of the z_k's, it then follows that

$$e^{-i\theta_j}\delta w_n = -\frac{m}{M}\sum_{k\neq n} e^{i\theta_{k-j}}\delta w_k.$$

Making this substitution for $e^{-i\theta_j}\delta w_n$ and an analogous substitution for $e^{i\theta_j}\delta\bar{w}_n$ in (18), we see that

$$\delta\ddot{w}_j = \omega^2\delta w_j - 2i\omega\delta\dot{w}_j + \frac{Gm}{2r^3}\sum_{k\neq n}\left(e^{i\theta_{k-j}}\delta w_k + 3e^{-i\theta_{k-j}}\delta\bar{w}_k\right) + \frac{GM}{2r^3}(\delta w_j + 3\delta\bar{w}_j)$$

$$-\frac{Gm}{2r^3}\,\frac{1}{8}\sum_{k\neq j,n}\frac{e^{i\theta_{k-j}}\delta w_k-\delta w_j-3e^{i\theta_{k-j}}(e^{-i\theta_{k-j}}\delta\bar{w}_k-\delta\bar{w}_j)}{\sin^3(|\theta_{k-j}|/2)}$$

$$+\frac{3}{4}\frac{Gm\mathcal{J}_2R^2}{r^5}\sum_{k\neq n}\left(3e^{i\theta_{k-j}}\delta w_k+5e^{-i\theta_{k-j}}\delta\bar{w}_k\right)+\frac{3}{4}\,\frac{GM\mathcal{J}_2R^2}{r^5}(3\delta w_j+5\delta\bar{w}_j). \quad (19)$$

5. Circulant Matrix

Switching to matrix notation, let W_j denote a shorthand for the column vector $\left[\begin{array}{cc} w_j & \bar{w}_j \end{array}\right]'$. In this notation, we see that (19) together with its conjugates can be written as

$$\frac{d}{dt}\left[\begin{array}{c}\delta W_0\\ \delta W_1\\ \vdots\\ \delta W_{n-1}\\ \delta\dot{W}_0\\ \delta\dot{W}_1\\ \vdots\\ \delta\dot{W}_{n-1}\end{array}\right]\approx\left[\begin{array}{cccc|cccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I\\ \hline D & N_1 & \cdots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ N_1 & N_2 & \cdots & D & & & & \Omega\end{array}\right]\left[\begin{array}{c}\delta W_0\\ \delta W_1\\ \vdots\\ \delta W_{n-1}\\ \delta\dot{W}_0\\ \delta\dot{W}_1\\ \vdots\\ \delta\dot{W}_{n-1}\end{array}\right], \quad (20)$$

where D, Ω, and the N_k's are 2×2 complex matrices given by

$$D = \omega^2\left[\begin{array}{cc}1&0\\0&1\end{array}\right]+\frac{Gm}{2r^3}\left[\begin{array}{cc}1+J_n/2+\frac{9}{2}\mathcal{J}_2\,(R/r)^2 & 3-3J_n/2+3I_n+\frac{15}{2}\mathcal{J}_2\,(R/r)^2\\ 3-3J_n/2+3I_n+\frac{15}{2}\mathcal{J}_2\,(R/r)^2 & 1+J_n/2+\frac{9}{2}\mathcal{J}_2\,(R/r)^2\end{array}\right]$$

$$+\frac{GM}{2r^3}\left[\begin{array}{cc}1+\frac{9}{2}\mathcal{J}_2\,(R/r)^2 & 3+\frac{15}{2}\mathcal{J}_2\,(R/r)^2\\ 3+\frac{15}{2}\mathcal{J}_2\,(R/r)^2 & 1+\frac{9}{2}\mathcal{J}_2\,(R/r)^2\end{array}\right]$$

$$N_k = \frac{Gm}{2r^3}\left[\begin{array}{cc}e^{i\theta_k}\left(1-J_{k,n}/2+\frac{9}{2}\mathcal{J}_2\,(R/r)^2\right) & 3e^{-i\theta_k}+3J_{k,n}/2+\frac{15}{2}e^{-i\theta_k}\mathcal{J}_2\,(R/r)^2\\ 3e^{i\theta_k}+3J_{k,n}/2+\frac{15}{2}e^{i\theta_k}\mathcal{J}_2\,(R/r)^2 & e^{-i\theta_k}\left(1-J_{k,n}/2+\frac{9}{2}\mathcal{J}_2\,(R/r)^2\right)\end{array}\right]$$

$$\Omega = 2i\omega\left[\begin{array}{cc}-1&0\\0&1\end{array}\right],$$

and where

$$
\begin{aligned}
I_{k,n} &= \frac{1}{4\sin\left(\pi|k|/n\right)} \\
J_{k,n} &= \frac{1}{4\sin^3\left(\pi|k|/n\right)}
\end{aligned}
$$

and

$$
I_n = \sum_{k=1}^{n-1} I_{k,n} \approx \frac{1}{2\pi}\, n \sum_{k=1}^{(n-1)/2} \frac{1}{k} \approx \frac{1}{2\pi}\, n\log(n/2) \tag{21}
$$

$$
J_n = \sum_{k=1}^{n-1} J_{k,n} \approx \frac{1}{2\pi^3} n^3 \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{n^3}{2\pi^3}\,\zeta(3) = 0.01938\, n^3. \tag{22}
$$

Here, the symbol $\approx$ is used to indicate asymptotic agreement. That is, $a_n \approx b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$ and $\zeta(3)$ denotes the value of the Riemann zeta function at 3. This constant is known as *Apéry's constant* (see, e.g., Arfken (1985)).

Finally, note that in deriving (20) from (19) we used the following identity

$$
\sum_{k=1}^{n-1} \frac{e^{i\theta_k}}{\sin^3|\theta_k|/2} = 4J_n - 8I_n. \tag{23}
$$

Let A denote the matrix in (20). We need to find the eigenvalues of A and derive necessary and sufficient conditions under which none of them have a positive real part. At this point we could resort to numerical computation to bracket a threshold for stability by doing a binary search to find the largest value of m/M for which none of the eigenvalues have positive real part. We did such a search for some values of n. The results are shown in Table **??**.

The eigenvalues are complex numbers for which there are nontrivial solutions to

$$
\left[\begin{array}{cccc|cccc}
 & & & & I & & & \\
 & & & & & I & & \\
 & & & & & & \ddots & \\
 & & & & & & & I \\
\hline
D & N_1 & \cdots & N_{n-1} & \Omega & & & \\
N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\
\vdots & \vdots & & \vdots & & & \ddots & \\
N_1 & N_2 & \cdots & D & & & & \Omega
\end{array}\right]
\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}
= \lambda
\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}. \tag{24}
$$

The first group of equations (above the line) can be used to eliminate the “derivative” variables from the second set. That is,

$$\begin{bmatrix} \delta\dot{W}_0 \\ \delta\dot{W}_1 \\ \vdots \\ \delta\dot{W}_{n-1} \end{bmatrix} = \lambda \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix}$$

and therefore

$$\begin{bmatrix} D & N_1 & \cdots & N_{n-1} \\ N_{n-1} & D & \cdots & N_{n-2} \\ \vdots & \vdots & & \vdots \\ N_1 & N_2 & \cdots & D \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} + \lambda \begin{bmatrix} \Omega & & & \\ & \Omega & & \\ & & \ddots & \\ & & & \Omega \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \lambda^2 \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix}. \tag{25}$$

The matrix on the left-hand side is called a *block circulant matrix.* Much is known about such matrices. In particular, it is easy to find the eigenvectors of such matrices. For general properties of block circulant matrices, see Tee (2005).

Let ρ denote an n-th root of unity (i.e., $\rho = e^{2\pi ij/n}$ for some $j = 0, 1, \ldots, n-1$) and let ξ be an arbitrary complex 2-vector. We look for solutions of the form

$$\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \begin{bmatrix} \xi \\ \rho\xi \\ \vdots \\ \rho^{n-1}\xi \end{bmatrix}.$$

Substituting such a guess into (25), we see that each of the n rows reduce to one and the same thing

$$\left(D + \rho N_1 + \ldots + \rho^{n-1} N_{n-1}\right) \xi + \lambda\Omega\xi = \lambda^2\xi.$$

There are nontrivial solutions to this 2×2 system if and only if

$$\det(D + \rho N_1 + \ldots + \rho^{n-1} N_{n-1} + \lambda\Omega - \lambda^2 I) = 0.$$

For each root of unity, ρ, there are four values of λ that solve this equation (counting multiplicites). That makes a total of $4n$ eigenvalues and therefore provides all eigenvalues for the full system (24).

6. Explicit Expression for $\sum_{k=1}^{n-1}\rho^k N_k$

In order to compute the eigenvalues, it is essential that we compute $\sum_{k=1}^{n-1}\rho^k N_k$ as explicitly as possible. To this end, we note the following reduction and new definition:

$$\begin{aligned}\sum_{k=1}^{n-1}\rho^k J_{k,n} &= \frac{1}{4}\sum_{k=1}^{n-1}\frac{e^{2\pi ijk/n}}{\sin^3(\theta_k/2)} \\ &= \frac{1}{4}\sum_{k=1}^{n-1}\frac{\cos(j\theta_k)}{\sin^3(\theta_k/2)} \\ &=: \tilde{J}_{j,n}.\end{aligned}\tag{26}$$

Similarly,

$$\sum_{k=1}^{n-1}\rho^k e^{i\theta_k} J_{k,n} = \tilde{J}_{j+1,n}\tag{27}$$

and

$$\sum_{k=1}^{n-1}\rho^k e^{-i\theta_k} J_{k,n} = \tilde{J}_{j-1,n}.\tag{28}$$

Also we compute

$$\sum_{k=1}^{n-1}\rho^k e^{i\theta_k} = \sum_{k=1}^{n-1}e^{ij\theta_k}e^{i\theta_k} = \sum_{k=1}^{n-1}e^{i(j+1)\theta_k} = \begin{cases} n-1 & \text{for } j=n-1 \\ -1 & \text{otherwise}\end{cases}\tag{29}$$

and

$$\sum_{k=1}^{n-1}\rho^k e^{-i\theta_k} = \begin{cases} n-1 & \text{for } j=1 \\ -1 & \text{otherwise .}\end{cases}\tag{30}$$

Substituting the definition of N_k into $\sum_{k=1}^{n-1}\rho^k N_k$ and making use of (26)-(30), we get

$$\begin{aligned}\sum_{k=1}^{n-1}\rho^k N_k &= \frac{Gm}{2r^3}\begin{bmatrix} -1+n\delta_{j=n-1}-\frac{1}{2}\tilde{J}_{j+1,n} & -3+3n\delta_{j=1}+\frac{3}{2}\tilde{J}_{j,n} \\ -3+3n\delta_{j=n-1}+\frac{3}{2}\tilde{J}_{j,n} & -1+n\delta_{j=1}-\frac{1}{2}\tilde{J}_{j-1,n}\end{bmatrix} \\ &\quad -\frac{Gm}{4r^3}\mathcal{J}_2\,(R/r)^2\begin{bmatrix} 9 & 15 \\ 15 & 9\end{bmatrix},\end{aligned}\tag{31}$$

where $\delta_{j=k}$ denotes the Kronecker delta (i.e., one when $j=k$ and zero otherwise).

Henceforth, we assume that $j \neq 1, n-1$. Combining the expression defining D with (31), we get

$$D+\sum_{k=1}^{n-1}\rho^k N_k = \omega^2\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} + \frac{Gm}{2r^3}\begin{bmatrix} (J_n-\tilde{J}_{j+1,n})/2 & -3(J_n-\tilde{J}_{j,n})/2+3I_n \\ -3(J_n-\tilde{J}_{j,n})/2+3I_n & +(J_n-\tilde{J}_{j-1,n})/2\end{bmatrix}$$

$$+\frac{GM}{2r^3}\left[\begin{array}{cc} 1 & 3 \\ 3 & 1 \end{array}\right] + \frac{GM}{4r^3}\mathcal{J}_2\,(R/r)^2\left[\begin{array}{cc} 9 & 15 \\ 15 & 9 \end{array}\right] \tag{32}$$

Using (8) to eliminate ω, we see that

$$D + \sum_{k=1}^{n-1}\rho^k N_k = \frac{Gm}{2r^3}\left[\begin{array}{cc} (J_n - \tilde{J}_{j+1,n})/2 + 2I_n & -3(J_n - \tilde{J}_{j,n})/2 + 3I_n \\ -3(J_n - \tilde{J}_{j,n})/2 + 3I_n & +(J_n - \tilde{J}_{j-1,n})/2 + 2I_n \end{array}\right]$$

$$+\frac{GM}{2r^3}\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right)\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]. \tag{33}$$

7. Large n

When n is large, $\tilde{J}_{n/2\pm1,n} \approx \tilde{J}_{n/2,n}$ and $J_n \gg I_n$. Furthermore,

$$\tilde{J}_{n/2,n} \approx \frac{1}{2}\sum_{k=1}^{n/2}\frac{(-1)^k}{\sin^3(k\pi/n)} \approx \frac{n^3}{2\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}$$

$$= -\frac{3}{4}\frac{n^3}{2\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3} \approx -\frac{3}{4}\,\frac{1}{2}\sum_{k=1}^{n/2}\frac{1}{\sin^3(k\pi/n)} \approx -\frac{3}{4}J_n.$$

Hence,

$$D + \sum_{k=1}^{n-1}\rho^k N_k \approx \bar{m}\frac{7}{8}J_n\left[\begin{array}{cc} 1 & -3 \\ -3 & 1 \end{array}\right] + \bar{M}\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right)\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right], \tag{34}$$

where

$$\bar{m} = \frac{Gm}{2r^3} \qquad \text{and} \qquad \bar{M} = \frac{GM}{2r^3}$$

8. Solving $\det\left(D + \sum_{k=1}^{n-1}\rho^k N_k + \lambda\Omega - \lambda^2 I\right) = 0.$

From (34), we see that

$$\det\left(D + \sum_{k=1}^{n-1}\rho^k N_k + \lambda\Omega - \lambda^2 I\right) = \left|\begin{array}{cc} A - 2i\lambda\omega - \lambda^2 & B \\ B & A + 2i\lambda\omega - \lambda^2 \end{array}\right|$$

$$= \lambda^4 + (4\omega^2 - 2A)\lambda^2 + A^2 - B^2. \tag{35}$$

where

$$A = \frac{7}{8}\bar{m}J_n + \bar{M}\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right) \tag{36}$$

$$B = -\frac{21}{8}\bar{m}J_n + \bar{M}\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right). \tag{37}$$

For convenience we rewrite this biquadratic polynomial in λ as

$$\det\left(D + \sum_{k=1}^{n-1}\rho^k N_k + \lambda\Omega - \lambda^2 I\right) = \lambda^4 + b\lambda^2 + c = 0 \tag{38}$$

where

$$\begin{aligned} b &= 4\omega^2 - 2A \\ &= \left(8I_n - \frac{7}{4}J_n\right)\bar{m} + \left(2 - 3\mathcal{J}_2\,(R/r)^2\right)\bar{M} \\ &\approx -\frac{7}{4}J_n\bar{m} + \left(2 - 3\mathcal{J}_2\,(R/r)^2\right)\bar{M} \end{aligned} \tag{39}$$

and

$$\begin{aligned} c &= A^2 - B^2 \\ &= 7\bar{m}J_n\left(-\frac{7}{8}\bar{m}J_n + 7\bar{M}\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right)\right). \end{aligned} \tag{40}$$

The eigenvalues are found by setting the right-hand side to zero. From the quadratic formula, we see that

$$\lambda^2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \tag{41}$$

Clearly, the four solutions to the biquadratic equation will have nonnegative real part if and only if both values of λ^2 lie on the nonpositive real axis. For this, it is necessary and sufficient that

$$b \geq 0 \tag{42}$$

$$c \geq 0 \tag{43}$$

$$b^2 - 4c \geq 0. \tag{44}$$

In other words, we need

$$\frac{7}{4} \leq \left(2 - 3\mathcal{J}_2\,(R/r)^2\right)\gamma \tag{45}$$

$$-\frac{7}{8} + \left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right)\gamma \geq 0 \tag{46}$$

$$\left(\frac{7}{4} - \left(2 - 3\mathcal{J}_2\,(R/r)^2\right)\gamma\right)^2 \geq 28\left(-\frac{7}{8} + \gamma\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right)\right), \tag{47}$$

where

$$\gamma = \frac{\bar{M}}{\bar{m}J_n} = \frac{M}{mJ_n}. \tag{48}$$

The first two inequalities are linear in γ whereas the third one is quadratic. All of them are easy to solve:

$$\gamma \geq \frac{7}{4\left(2 - 3\mathcal{J}_2\,(R/r)^2\right)} \tag{49}$$

$$\gamma \geq \frac{7}{8\left(3 + \frac{15}{2}\mathcal{J}_2\,(R/r)^2\right)} \tag{50}$$

$$\gamma \geq \frac{7}{8}\frac{13 - \frac{57}{2}\mathcal{J}_2\left(\frac{R}{r}\right)^2 + \sqrt{\left(13 - \frac{57}{2}\mathcal{J}_2\left(\frac{R}{r}\right)^2\right)^2 - 9\left(1 - \frac{3}{2}\mathcal{J}_2\left(\frac{R}{r}\right)^2\right)^2}}{1 - \frac{3}{2}\mathcal{J}_2\left(\frac{R}{r}\right)^2}, \tag{51}$$

For a sphere, $\mathcal{J}_2 = 0$. For a flat disk, $\mathcal{J}_2 = 1/4$. For the ring to be outside the central body we have $R \leq r$. Hence, we many assume that $0 \leq \mathcal{J}_2\,(R/r)^2 \leq 1/4$. Over this range of values, it is easy to check that inequality (51) is more constraining than the previous two. Assuming that the four eigenvalues given by picking $j = n/2$ determine stability of the entire system, as was rigorously proved for non-oblate central bodies by many previously (see, e.g., Vanderbei and Kolemen (2007)), it follows that the ring system is linearly stable if and only if γ satisfies (51).

9. A Model for a System of Rings

Real ring systems, such as the one around Saturn, consist not just of a number of bodies at a single radius but rather as a swarm of bodies distributed over a broad, nearly-continuous range of radii out to the Roche limit. Modelling such a system at the level of precision we have employed for a single ring seems highly intractible. In fact, even just two rings is probably an intractible problem at that level of detail. But, I contend that any instability of the particles at a given radius would almost certainly be produced by interactions between bodies essentially at this one radius. In other words, it should be possible to investigate stability of broad ring systems by studying a single-radius ring where the effect of the bodies at smaller and larger radii can be modeled in aggregate. In fact, we lump the effect of the bodies at other radii with our central mass and consider a new model for our central mass. It is no longer considered as a simple oblate spheroid. Instead, we model the central mass as having two parts each of which is an oblate spheroid. The first part is the very massive, slightly oblate spheroid representing the central body itself and the second part is the much less massive, but completely flat (i.e., $\mathcal{J}_2 = 1/4$) disk representing the rings at all radii except the one under consideration. In other words, we assume that the central gravitational potential is given as

$$\begin{aligned} V &= -\frac{GM_{\rm S}}{r} - \frac{GM_{\rm S}}{r}\mathcal{J}_{2,\rm S}\left(\frac{R_{\rm S}}{r}\right)^2 \frac{1-3\sin^2\phi}{2} \\ & \quad -\frac{GM_{\rm R}}{r} - \frac{GM_{\rm R}}{r}\mathcal{J}_{2,\rm R}\left(\frac{R_{\rm R}}{r}\right)^2 \frac{1-3\sin^2\phi}{2} \end{aligned} \tag{52}$$

where ϕ denotes latitude and, as always, r denotes radius. Of course, $M_{\rm S}$, $\mathcal{J}_{2,\rm S}$, and $R_{\rm S}$ denote the mass, oblateness, and equatorial radius of the slightly oblate central body and $M_{\rm R}$, $\mathcal{J}_{2,\rm R}$, and $R_{\rm R}$ denote the mass, oblateness, and equatorial radius of the smaller-mass, flat ring system. Because this oblate component is assumed to be completely flat, i.e. a disk, we will later substitute $\mathcal{J}_{2,\rm R} = 1/4$. We will also later substitute the Roche limit for $R_{\rm R}$.

The stability analysis of the previous sections can be repeated almost verbatim using this slightly more elaborate gravitational field. The result is that a ring of n particles at radius r is linearly stable if and only if

$$\gamma \geq \frac{7}{8}\frac{13 - \frac{57}{2}\mathcal{B} + \sqrt{\left(13-\frac{57}{2}\mathcal{B}\right)^2 - 9\left(1-\frac{3}{2}\mathcal{B}\right)^2}}{1-\frac{3}{2}\mathcal{B}}, \tag{53}$$

where

$$\mathcal{B} = \mu_{\rm S}\mathcal{J}_{2,\rm S}\left(\frac{R_{\rm S}}{r}\right)^2 + \mu_{\rm R}\mathcal{J}_{2,\rm R}\left(\frac{R_{\rm R}}{r}\right)^2 \tag{54}$$

and

$$\mu_{\rm S} = M_{\rm S}/(M_{\rm S}+M_{\rm R}) \quad \text{and} \quad \mu_{\rm R} = M_{\rm R}/(M_{\rm S}+M_{\rm R}). \tag{55}$$

For Saturn, $\mu_{\rm R} \approx 6.2 \times 10^{-8}$, $\mathcal{J}_{2,\rm S} \approx 0.09786$, and $R_{\rm S} \approx 60220$ km. Furthermore, for a flat ring, $\mathcal{J}_{2,\rm R} = 1/4$. For $R_{\rm R}$ we use, say, the radius of Saturn's F ring ($\approx 140,180$ km). With these parameters, we see that the first term in (54) dominates the second term by six orders of magnitude. In other words, the stability of a ring at any particular radius is affected more by the oblateness of Saturn than it is by the ring bodies at other radii. But, of course, it is most affected by other bodies at the same radius.

10. Ring Density

Suppose that the linear density of the boulders is λ. That is, λ is the ratio of the diameter of one boulder to the separation between the centers of two adjacent boulders. Then the diameter of a single boulder is $\lambda(2\pi r/n)$. Hence, the volume of a single boulder is

$(4\pi/3)(\lambda\pi r/n)^3$. Let δ denote the density of a boulder. Then the mass of a single boulder is $(4\pi/3)(\lambda\pi r/n)^3\delta$. If we assume that the density of Earth is about 8 times that of a boulder (Earth's density is 5.5 and Saturn's moons have a density of about 0.7 being composed of porous water-ice), then we have

$$\delta = \frac{1}{8}\frac{m_E}{(4\pi/3)r_E^3},$$

where m_E denotes the mass of Earth and r_E denotes its radius. Combining all of these factors, we see that an upper bound on the linear density of boulders at radius r is

$$\lambda \leq \lambda_c = \left(8\frac{M/m_E}{0.01938\gamma_c}\right)^{1/3}\frac{r_E}{\pi r},$$

where

$$\gamma_c = \frac{7}{8}\frac{13 - \frac{57}{2}\mathcal{B} + \sqrt{\left(13 - \frac{57}{2}\mathcal{B}\right)^2 - 9\left(1 - \frac{3}{2}\mathcal{B}\right)^2}}{1 - \frac{3}{2}\mathcal{J}_2\mathcal{B}}$$

and $\mathcal{B}$ is given by (54). Figure 2 shows a plot of λ_c as a function of radius for Saturn. The values range between 0.19 (at the radius of the F ring) to 0.42 at the surface of Saturn. In other words, the density of the rings is predicted by this model to be fairly constant from the surface of Saturn out to the Roche limit ($\approx$ 147000 km). Of course, Saturn's rings don't actually extend to the surface of Saturn. Electromagnetic interactions with Saturn itself disrupts any ring below a certain minimum radius. But, from this minimum radius to the Roche limit, our computed value is fairly constant and is consistent with Saturn's measured optical density, which ranges from 0.05 to 2.5.

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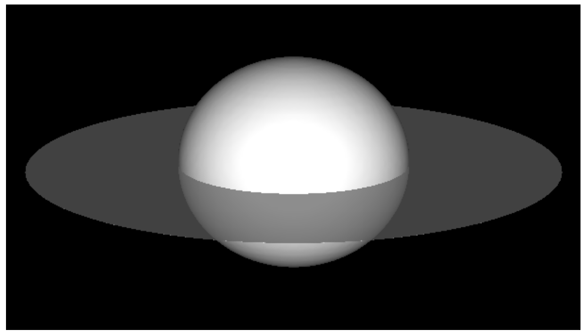

Fig. 1.— The gravitational potential of the “central mass” corresponds to this geometry.

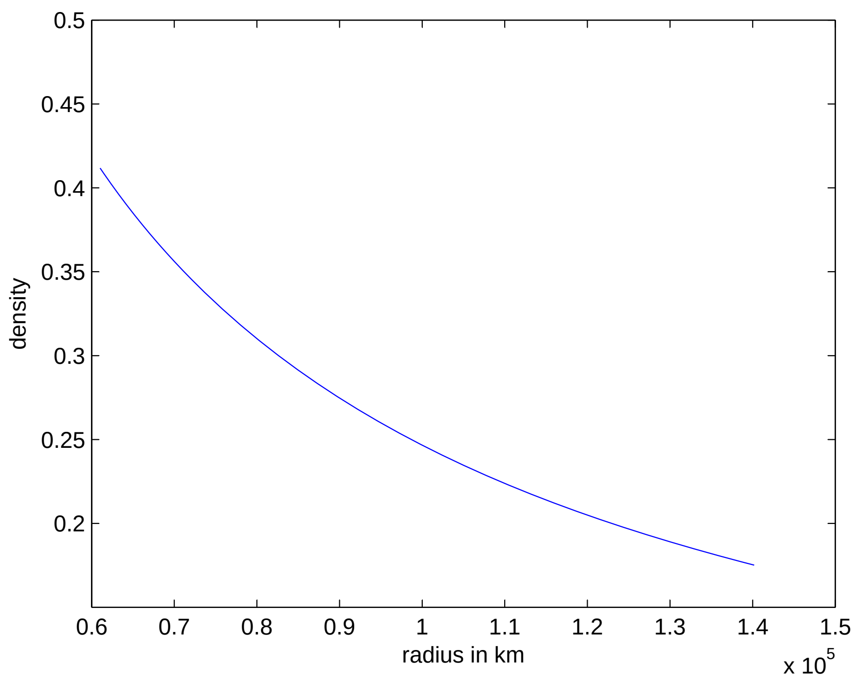

Fig. 2.— Linear density λ_c as a function of radius.