# Horsing Around on Saturn

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## ABSTRACT

Based on simple statistical mechanical models, the prevailing view of Saturn's rings is that they are unstable and must therefore have been formed rather recently. In this paper, we argue that Saturn's rings and inner moons are in much more stable orbits than previously thought and therefore that they likely formed together as part of the initial formation of the solar system. To make this argument, we give a detailed description of so-called *horseshoe orbits* and show that this horseshoeing phenomenon greatly stabilizes the rings of Saturn.

This paper is part of a collaborative effort with E. Belbruno and J.R. Gott III. For a description of their part of the work, see their papers in these proceedings.

## 1. Introduction

The currently accepted view (see, e.g., Goldreich and Tremaine (1982)) of the formation of Saturn's rings is that a moon-sized object wandered inside Saturn's Roche limit and was torn apart by tidal forces. The resulting array of remnant masses was dispersed and formed the rings we see today. In this model, the system dynamics are treated as a turbulent flow with no explicit mention of the law of gravitation. The ring shape is presumed to be maintained by the coralling effect of a few of the inner moons acting as so-called *shepherds*.

Of course, a simpler explanation would be that the rings formed along with the planets and their moons as they condensed out of the solar nebula.

The reason given in support of the first, seemingly unlikely, scenario is that the rings are unstable, their persistance time being measured in the hundreds of millions of years, but not billions. Hence, it is supposed that the rings could not have formed at the time of the formation of the Solar System. The rationale for claiming that the rings are unstable is that particle collisions cause a slow decrease in energy resulting in the particles slowly migrating inward and crashing into Saturn itself.

In this paper, we argue that collisions are more rare than a simple "gas-dynamics" model would suggest. Hence, we claim that the rings were likely formed together with Saturn and the other planets from the original solar nebula. Collisions are rare because of an effective repelling force which is best understood by considering simple horseshoe orbits. Hence, much of this paper is devoted to a discussion of the dynamics behind horseshoe orbits.

This paper is the result of a fruitful collaborative effort with E. Belbruno and J.R. Gott III. In fact, this paper is a natural In fact, this work on Saturn and horseshoe orbits is a natural outgrowth of their recent paper on the origin of Earth's Moon—see Belbruno and Gott (2005). The reader is also encouraged to look at Howard (2005) which shows how the Belbruno-Gott model for our moon can be applied to Saturn's moons.

## 2. Shouldn't Ring Systems be Unstable?

We start by discussing a situation that is arguably far too simple. We do so because it helps illustrate the enormous importance of a massive central body (Saturn).

Consider a large collection of masses all having about the same mass. In fact, for simplicity, assume that they all have exactly the same mass. Think of these masses as particles making up the rings of Saturn. It is well known that it is difficult to find stable solutions to the equal-mass $n$-body problem for $n \geq 3$. However, for $n = 2$ (i.e., the 2-body problem), the problem is trivial: if the two bodies start out orbiting their center of mass in an elliptical orbit, then this solution is stable and periodic for all time. A special case of this scenario occurs when the elliptical orbits are in fact circular. In this case, the two bodies exhibit circular orbits about the center of mass which lies exactly halfway between them (because the masses are the same). In some sense this is a trivial example of a ring system where the ring consists of just two particles and the center planet is missing. Now, to make this more like a real ring system, let's set $n \geq 3$ and distribute the masses uniformly in a circle around their center of mass and give them appropriate initial velocities so that they ought to exhibit circular motion about the center of mass. By simple symmetry arguments, it is easy to see that this must be a valid solution to the $n$-body problem. But, as even the most elementary implementation of an $n$-body simulator will quickly reveal, such a system is unstable for all $n \geq 3$. The reason is very simple. Imagine that one of the bodies gets ahead in its orbit by the tiniest amount. Then the pull from the body that is ahead of it increases slightly (because it is now closer) and the pull from the body that lags it decreases (for an analgous reason). Hence, the body draws even further ahead and we see that the system is unstable. Figure 1 shows some snapshots illustrating this instability for the 3-body problem.

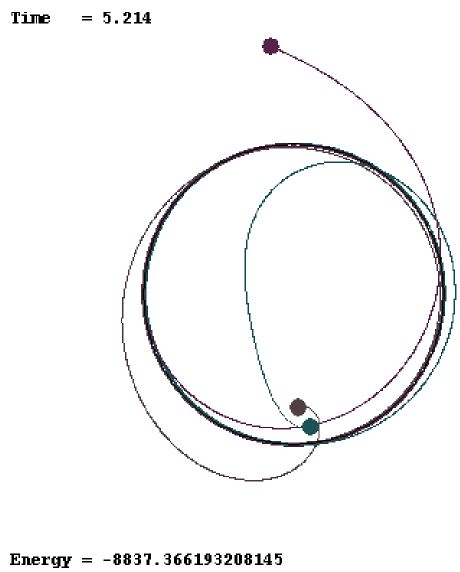


Fig. 1.— For 3 or more bodies, the "circular" system is unstable. On the left is a snapshot taken just after the simulation was started. On the right is a plot showing the paths of the three bodies over several orbits. All starts out well, but eventually instability becomes apparent. To run the Java applet that generated these plots, visit http://www.princeton.edu/∼rvdb/JAVA/astro/saturnTalk/Lagrange3.html and click the start button.

## 3. A Large Central Mass Stabilizes

The instability discussed in the previous section would seem to spell bad news for the stability of ring systems. However, a large central mass, such as Saturn itself, has an enormous stabilizing effect. This effect is well understood in the context of Lagrange points. It is known that systems consisting of a large central mass and two other smaller equal masses started in circular orbits with one of the small masses leading the other by 60° are stable provided that the mass ratios are large enough (the central mass must be about 25 times greater than the other two masses). Not only is this understood mathematically, it is also observed. For example, there are two large agglomerations of asteroids, known as the *Trojan asteroids*, orbiting the Sun at the same radius as Jupiter, one is 60° ahead and the other trails by 60°. They are seen in asteroid surveys and they are easy to simulate. An example simulation is shown in Figure 2.

A large central mass can stabilize not only a pair of bodies in circular orbits, but also pairs in eccentric orbits. One such example is shown in Figure 3. In that figure, the central body has the mass of Saturn and the two smaller bodies have the masses of Saturn's moons Janus and Epimetheus. The mass $m_\Sigma$ of Saturn is much larger than the masses $m_\mathrm{J}$ of Janus and $m_\mathrm{E}$ of Epimetheus:

$$\begin{aligned} m_\Sigma/m_\mathrm{J} &= 2.8 \times 10^8 \\ m_\Sigma/m_\mathrm{E} &= 1.0 \times 10^9. \end{aligned}$$

In the real Saturn-Janus-Epimetheus system, the radii of the two moons are almost identical at about 151000km from the center of Saturn. This puts these two moons just outside the Roche limit and just beyond the end of Saturn's rings. If the two moons are started at exactly the same radii with one leading and one lagging by 60°, then they remain at all times in this L4/L5 arrangement. This is true even if the two bodies are put on eccentric orbits.

## 4. Janus and Epimetheus: Two Horseshoeing Moons of Saturn

Janus and Epimetheus were chosen for the example in the previous section because these two moons of Saturn do, in fact, exhibit interesting orbits. Their true orbits are not the eccentric orbits illustrated in the previous section. Rather, their orbits are very close to circular with one body a scant 50km closer to Saturn than the other. Because the orbits are nearly circular, the body that is closer to Saturn has a slightly shorter period than the one that is further out. Hence, the closer body catches up to the further body once every four

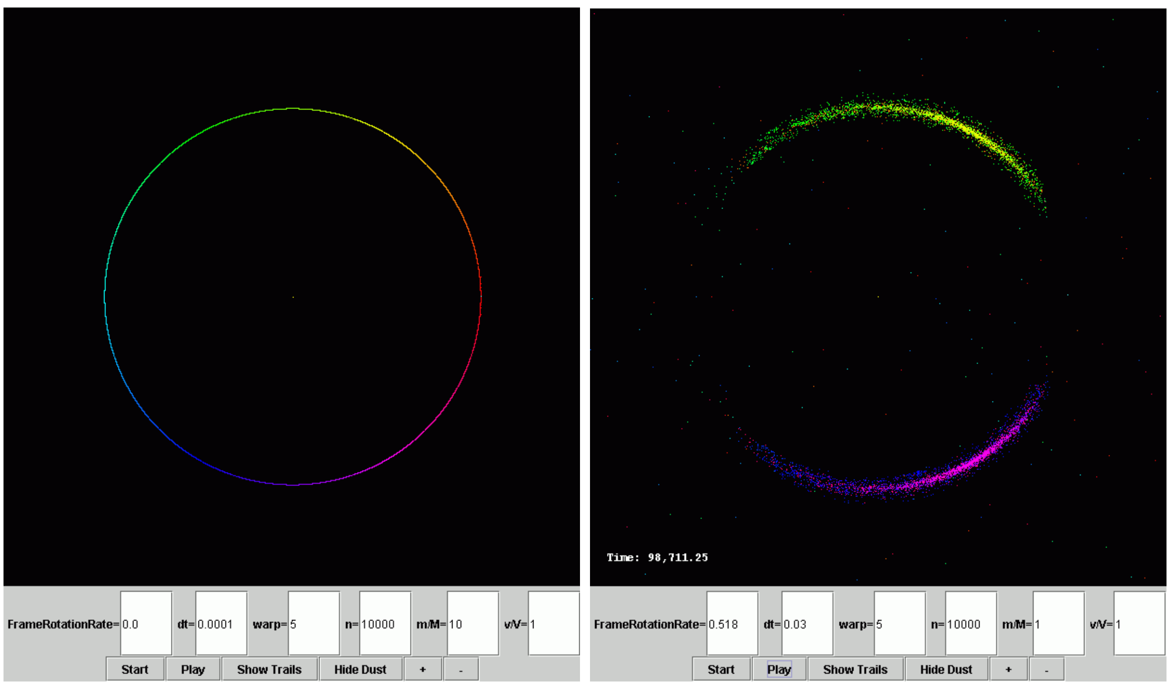


Fig. 2.— The Trojan asteroids illustrate how the L4 and L5 Lagrange points are stable. To run the Java applet that generated these plots, visit http://www.princeton.edu/*sim*rvdb/JAVA/astro/saturnTalk/StarSystem0.html and click the start button.

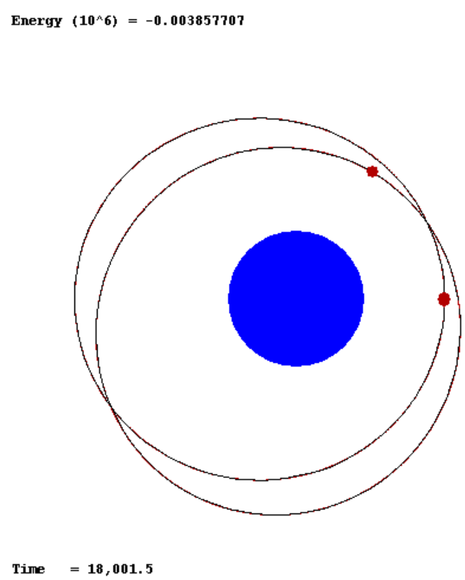


Fig. 3.— The 3-body problem. The central body has the mass of Saturn. The other two bodies have the masses of Saturn's moons Janus and Epimetheus. They are started in the L4/L5 position relative to each other but are given an initial velocity vector putting them into an eccentric orbit (an eccentricity of 0.2). This system is stable over many thousands of years, as this plot shows. To run the Java applet that generated these plots, visit http://www.princeton.edu/*sim*rvdb/JAVA/astro/saturnTalk/L4L5.html, adjust the eccentricity and click the start button.

years (individually, each body orbits Saturn about once every 17 hours). Suppose, for the sake of argument, that Epimetheus is closer to Saturn and catching up to Janus. Epimetheus does not overtake Janus. Instead as it starts to come up from behind it is accelerated forward by the pull of Janus. But, this acceleration lifts Epimetheus to a higher, slower orbit. In the same way, Janus decelerates and drops to a lower, faster orbit. They then begin moving away from each other with Janus now in the lower orbit and Epimetheus in the higher orbit. There roles have simply switched. This switching between which of the two moons is in the higher orbit and which is in the lower one continues indefinitely. It is called a *horseshoe orbit*. Such an orbit is depicted in Figure 4.

The true Saturn-Janus-Epimetheus system exhibits some features not yet accounted for. In particular, Janus has an eccentricity of $e = 0.004$ and Epimetheus has $e = 0.022$. Also, they have inclinations of 0.14 and 0.34 degrees, respectively. Even so, they horseshoe. And, it is even easy to simulate a horseshoeing system with these added features. See Figure 5.

## 5. How Might Such a Horseshoe System Form?

One might think from the previous section that Janus and Epimetheus represent a rather unusual circumstance. But nothing could be further from the truth. Among Saturn's 11 inner moons, there are three such systems.

Dione and Helene are at approximately the same orbital radii. They are not stuck in an L4/L5 configuration. Rather, at the moment, Helene leads Dione. How much it leads by varies from about 10° to about 180°. Simulations based on Horizon data (see Standish et al. (1983, 1995)) further suggest that in about 50 years they will start a full horseshoe type orbit oscillating back and forth getting as close as about 10° from each other at epochs of close approach.

Even more interesting is the trio consisting of Tethys, Telesto, and Calypso. With respect to Tethys, Telesto and Calypso lead and follow. But there is considerable oscillation from closest approaches that are about 20° to maximum angular separations of about 120°.

It was even recently discovered that Earth itself has a coorbital asteroid, 2002AA29, which forms a Sun-Earth-asteroid horseshoe system.

These examples lead us (Richard Gott, Ed Belbruno, and myself) to believe that horseshoeing coorbital systems are far from rare. In fact, they seem rather common and have attracted considerable interest (see, e.g., Nicholson et al. (1982); Salo and Yoder (1988); Naouni (1999); Murray and Dermott (1999); Llibre and Olle (2001)). So, we ask how do

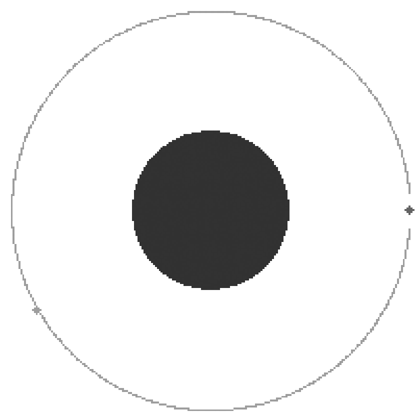

Fig. 4.— Saturn, Janus, and Epimetheus in an ideal horseshoe orbit. The bodies have their correct masses. The moons are placed at the correct orbital radius from Saturn (roughly 151000 km) and Saturn is shown in the correct scale. To limit confusion, the rings are not shown. The moons are shown much larger than their true size (about 90 km for Janus and 50km for Epimetheus) would dictate. The difference of 55 km in orbital radius is too small a difference to see at this scale. Nonetheless, even a simple integrator is able to simulate the horseshoe dynamic. As the simulator ran, Saturn was held fixed at the center of the coordinate system and the coordinate system was counterrotated to hold Janus fixed on the positive $x$-axis. In this rotating frame, Epimetheus horseshoed around sometimes approaching Janus from behind (below) in a lower orbit and at other times being overtaken by Janus. The figure shows the gap that determines closest approach which is about 13000 km. To run the Java applet that generated these plots, visit http://www.princeton.edu/*sim*rvdb/JAVA/astro/saturnTalk/idealJanusEpimetheus.html and click the start button.

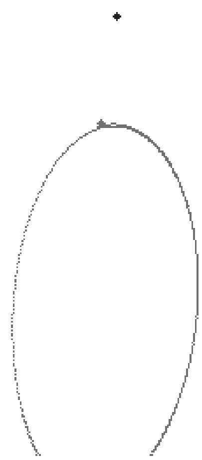

Fig. 5.— A simulation of Saturn, Janus, and Epimetheus using high-precision ephemeris data from JPL's Horizons system. The figure shows a closeup of an epoch of closest approach. As usual, the frame is counterrotated to hold Janus fixed on the positive $x$-axis and Saturn fixed at the origin (which is far off the left edge). To run the Java applet that generated this plot, visit http://www.princeton.edu/*sim*rvdb/JAVA/astro/saturnTalk/realJanusEpimetheus.html and click the start button.

they come about? Our thesis is that these bodies originally formed at essentially the same radius, perhaps even in an L4/L5 configuration. But, in many cases, the masses are so small they might have formed much further apart without feeling the influence of the other. But, once formed, they undoubtedly suffered various random perturbations. These random $\Delta v$'s would over time put the coorbiting moons at slightly differing radii and this would then set the horseshoeing in motion. To simulate this, we started a Saturn-Janus-Epimetheus system with Janus and Epimetheus at exactly the same radius from Saturn and configured in an L4/L5 arrangement (i.e., one leading the other by 60°). We then introduced small random $\Delta v$'s and watched the difference in orbital radius evolve over time. Figure 6 shows the result. Because we chose $\Delta v$'s that are both larger and more frequent than what one would expect in nature, the system evolved faster than one would expect. In fact, it reached breakout in just 11000 yrs. Nonetheless, this simulation seems rather indicative of what one would expect to see in nature over much longer time scales. And, the currently observed difference in orbital radius of about 55 km is just about what one would expect if one were to pick an epoch at random from this simulation.

Finally we mention that, at the time of breakout, the two moons could just barely miss colliding with each other and end up doing something rather chaotic and hard to predict (see Belbruno (2004) for a treatise on the complicated motions that can result). Or, if they miss by a lot, they might just end up as ordinary moons that are no longer coorbital and no longer horseshoeing. Or, they could hit each other. According to our simulations, when Janus and Epimetheus hit each other, their relative speeds are about 300 km/hr.

## 6. Toward a New Theory of the Rings of Saturn

The usual nomenclature of Lagrange points is biased toward the situation in which one of the orbiting masses is just a massless test particle, known as the restricted 3-body problem. In the case of Saturn's moons, the pairs (and triples) of moons have roughly equal masses. Hence, if Epimetheus is at the L4 point relative to Janus, then there could easily be a third moon at the L5 point relative to Janus. We see just this sort of thing with the Tethys-Telesto-Calypso system. In fact, one might expect that six coorbital moons to be distributed around Saturn at 60° intervals would be particularly stable. Such a system is indeed stable (at least according to numerical simulation). Furthermore, if these masses are given small random $\Delta v$'s, they begin a mutual horseshoe motion. That is, they remain essentially coorbital but they tend to repel each other as if cushioned by springs. This is shown in Figure 7. In that simulation, every $10^4 \Delta t$, each mass was given a random $\Delta v$ of magnitude $1.2 10^{-5}$km/s. Of course, the horseshoe bounce effect works with any number of

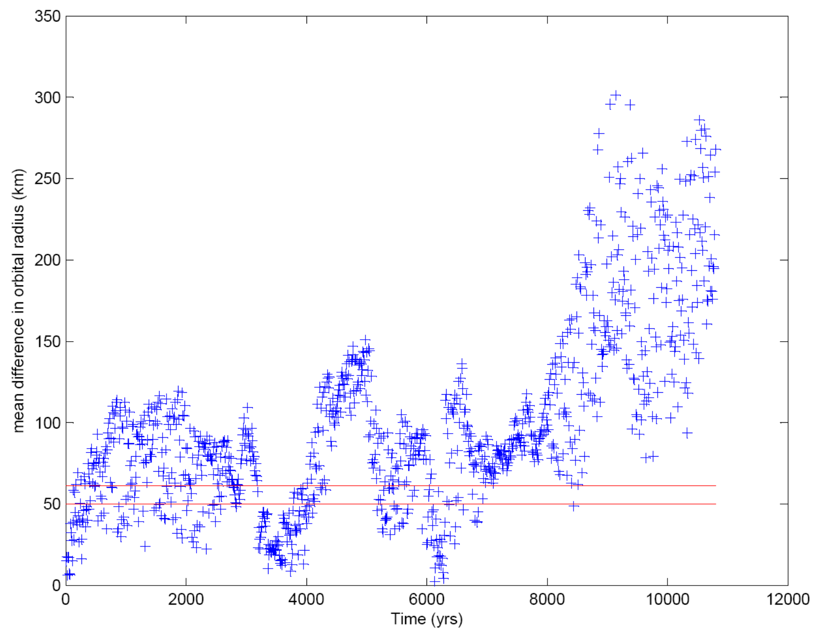


Fig. 6.— Starting from an L4/L5 configuration, Janus and Epimetheus were subjected to small random $\Delta v$'s. Over time, they began to horseshoe. Eventually, the difference in orbital radius became large enough to break out of the horseshoe orbit. This plot shows the evolution of the difference in orbital radius.

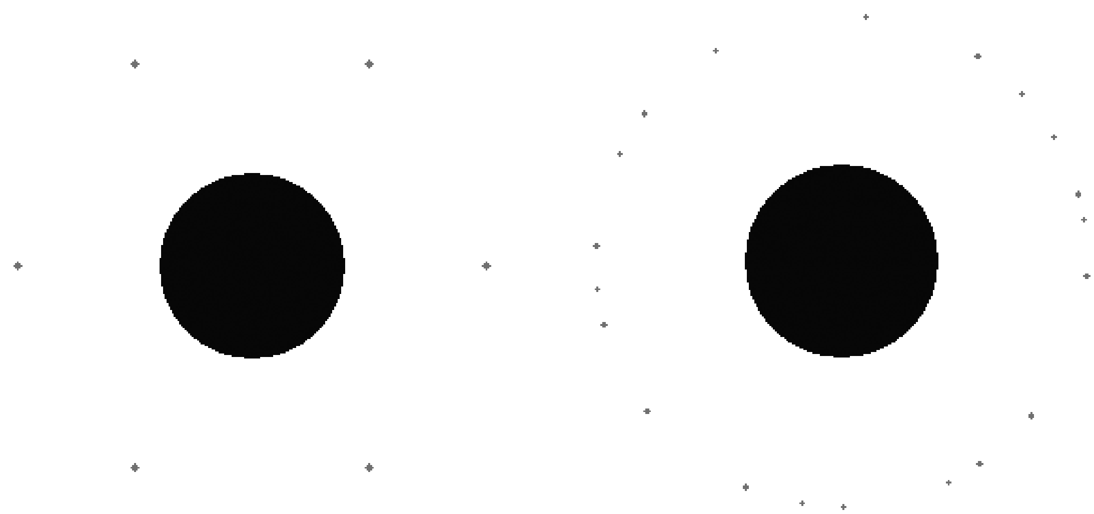

Fig. 7.— The left plot shows six Janus-sized masses distributed around Saturn. Initial distribution was uniform (as shown). Small random $\Delta v$'s were applied. The moons remain essentially coorbital. They do horseshoe-type bounces off from each other. The right-hand plot shows a 20-moon version after it has evolved for many years. To run the Java applet that generated this plot, visit http://www.princeton.edu/$sim$rvdb/JAVA/astro/saturnTalk/Janus6.html or http://www.princeton.edu/$sim$rvdb/JAVA/astro/saturnTalk/Janus20.html and click the start button.

masses. Figure 7 also shows a 20-moon scenario.

The number of masses can be increased to billions provided the total mass remains small. Hence, the horseshoe principle can be used to explain the dynamics of the rings of Saturn.

In the previous section, we mentioned that, when Janus and Epimetheus break out of a horseshoe orbit and collide, they do so with a relative speed in the hundreds of kilometers per hour. However, smaller bodies, such as the boulders that make up the rings of Saturn will hit with much less momentum. To see why, note that a marble orbits a bowling ball every 90 minutes if it is in low-bowling-ball orbit. That is, if we assume that a bowling ball has roughly same density as Earth, then an object in low orbit around the bowling ball has the same orbital period as an object in low-Earth-orbit. In other words, the period depends only on the density of the material. This is well-known and easy to show. A similar scaling effect works for horseshoe orbits. Indeed, if the masses are smaller, then the difference in orbital radius required for a horseshoe orbit is smaller. Janus and Epimetheus have diameters of about 100 km and a 55 km difference in orbital radius. Boulders of diameter 1 m would therefore need only about a 1 m difference in orbital radius to horseshoe. The difference in orbital velocity then is smaller by a proportional amount. Hence, a collision at 300 km/hr for moon-sized bodies translates to $300 \times 1/100,000$ km/hr, or 3 m/hr, for boulder-sized bodies. That is, they would just tap each other ever so gently. And, this is when collision occurs. The horseshoe effect introduces a gentle repelling force which makes these collisions quite uncommon. So our final thesis is that any initial chaotic motions in the planetary nebula at the time of formation of Saturn and its rings dampened out with the formation of the ice boulders that we now see. What remains is a myriad of boulders orbiting Saturn in almost perfect lock step with only gentle collisions and gentle repulsive forces keeping everything in a very stable configuration.

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