Frontiers of Stochastically Nondominated Portfolios

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(joint with A. Ruszczyński)

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ABSTRACT

1. The traditional (quadratic) Markowitz model produces portfolios that are stochastically dominated by portfolios not on the efficient frontier.

2. Replacing the quadratic risk measure with a mean absolute deviation (MAD) measure corrects this defect.

3. The MAD model can be formulated as a parametric linear programming problem (the risk parameter $\lambda$ is the parameter).

4. The *parametric* simplex method (described in detail in my book) can be used with $\lambda$ as the parameter of the parametric method.

5. Doing so, one finds ALL portfolios on the efficient frontier in roughly the same time as it takes to find just one portfolio (corresponding, say, to $\lambda = 0$).

6. The speedup is huge.
### Historical Data

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**Notation:** $R_j(t) = \text{return on investment } j \text{ in time period } t.$
The Ingredients: Risk and Reward

Raw Data:

\( R_j(t) \) = return on asset \( j \) in time period \( t \)

\( \Rightarrow \) Derived Data:

\[ \mu_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t) \]

\[ D_{tj} = R_j(t) - \mu_j. \]

Decision Variables:

\( x_j = \) fraction of portfolio to invest in asset \( j \)

\[ R(x) = \sum_j x_j R_j \]

Decision Criteria:

\[ \mu(x) = \sum_j \mu_j x_j \]

\[ \rho_2(x) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j D_{tj} x_j \right)^2 \]

\[ \rho_1(x) = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j D_{tj} x_j \right| \]
3. Quadratic Markowitz Problem

\[
\begin{align*}
\text{maximize} & \quad \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^T \left( \sum_j D_{tj} x_j \right)^2 \\
\text{subject to} & \quad \sum_j x_j = 1 \\
& \quad x_j \geq 0 \quad \text{for all investments } j
\end{align*}
\]

$\lambda$ is the risk parameter.
4. MAD Markowitz Problem

\[
\begin{align*}
\text{maximize} & \quad \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j D_{tj} x_j \right| \\
\text{subject to} & \quad \sum_j x_j = 1 \\
& \quad x_j \geq 0 \quad \text{for all investments } j
\end{align*}
\]

Not a linear programming problem. But it’s easy to convert.
Stochastic Dominance

Given: two random variables, \( V \) and \( S \).

- **First order stochastic dominance**: \( V \succeq_1 S \) means
  \[
  F_V(\eta) \leq F_S(\eta), \quad \text{for all } \eta \in \mathbb{R},
  \]
  where \( F_V \) and \( F_S \) denote the cumulative distribution functions of \( V \) and \( S \), respectively:
  \[
  F_V(\eta) = \mathbb{P}\{V \leq \eta\} \quad \text{for } \eta \in \mathbb{R}.
  \]

- **Second order stochastic dominance**: \( V \succeq_2 S \) means
  \[
  \int_{-\infty}^{\eta} F_V(\xi)\,d\xi \leq \int_{-\infty}^{\eta} F_S(\xi)\,d\xi, \quad \text{for all } \eta \in \mathbb{R}.
  \]

Henceforth, the term *stochastic dominance* without qualifier will refer to *second order* dominance.
Stochastic Dominance and Utility Theory

Second order stochastic dominance characterizes those random variables that every risk averse decision maker would prefer to a given random variable:

**Theorem** Random variable $V$ stochastically dominates random variable $S$ if and only if $\mathbb{E}(U(V)) \geq \mathbb{E}(U(S))$ for every increasing concave function $U(\cdot)$.

**Theorem** There are optimal solutions to the quadratic Markowitz model that are stochastically dominated by other (non-optimal) portfolios.

**Theorem** In the MAD Markowitz model, optimal portfolios are not stochastically dominated at least for all $\lambda \geq 2$. 
maximize \[ \lambda \sum_{j} \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t \]

subject to \[-y_t \leq \sum_{j} D_{tj} x_j \leq y_t \quad \text{for all times } t\]

\[ \sum_{j} x_j = 1 \]

\[ x_j \geq 0 \quad \text{for all investments } j \]

\[ y_t \geq 0 \quad \text{for all times } t \]
Adding Slack Variables $w_t^+$ and $w_t^-$

maximize \[ \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t \]

subject to \[ -y_t - \sum_j D_{tj} x_j + w_t^- = 0 \quad \text{for all times } t \]
\[ -y_t + \sum_j D_{tj} x_j + w_t^+ = 0 \quad \text{for all times } t \]
\[ \sum_j x_j = 1 \]
\[ x_j \geq 0 \quad \text{for all investments } j \]
\[ y_t, w_t^-, w_t^+ \geq 0 \quad \text{for all times } t \]
The Solution for Large $\lambda$

Varying the risk parameter $0 \leq \lambda < \infty$ produces the efficient frontier.

Large values of $\lambda$ favor reward whereas small values favor minimizing risk.

Beyond some finite threshold value for $\lambda$, the optimal solution will be a portfolio consisting of just one asset—the asset $j^*$ with the largest average return:

$$\mu_{j^*} \geq \mu_j \quad \text{for all } j.$$

It’s easy to identify basic vs. nonbasic variables:

- Variable $x_{j^*}$ is basic whereas the remaining $x_j$’s are nonbasic.
- All of the $y_t$’s are basic.
- If $D_{tj^*} > 0$, then $w_t^-$ is basic and $w_t^+$ is nonbasic. Otherwise, it is switched.
The Basic Optimal Solution for Large $\lambda$

Let

$$T^+ = \{ t : D_{tj^*} > 0 \}, \quad T^- = \{ t : D_{tj^*} < 0 \}, \quad \text{and} \quad \epsilon_t = \begin{cases} 1, & \text{for } t \in T^+ \\ -1, & \text{for } t \in T^- \end{cases}$$

It’s tedious, but here’s the optimal dictionary:

$$\zeta = \frac{1}{T} \sum_{t=1}^{T} \epsilon_t D_{tj^*} + \frac{1}{T} \sum_{j \neq j^*} \sum_{t=1}^{T} \epsilon_t (D_{tj} - D_{tj^*}) x_j + \lambda \mu_{j^*} + \lambda \sum_{j \neq j^*} (\mu_j - \mu_{j^*}) x_j$$

$$y_t = -D_{tj^*} - \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^- \quad t \in T^-$$

$$w_t^- = 2D_{tj^*} + 2 \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^+ \quad t \in T^+$$

$$y_t = D_{tj^*} + \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^+ \quad t \in T^+$$

$$w_t^+ = -2D_{tj^*} - 2 \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j + w_t^- \quad t \in T^-$$

$$x_{j^*} = 1 - \sum_{j \neq j^*} x_j$$
**Efficient Frontier**

Varying risk parameter $\lambda$ produces the so-called *efficient frontier*.

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<th>NASDAQ Comp.</th>
<th>Wilshire 5000</th>
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Daily Returns for 12 Years on 719 Assets

Click here for an expanded browser view.
Computing the Efficient Frontier

Using a reasonably efficient code for the parametric simplex method (simpo), it took 22,000 pivots and 1.5 hours to solve for one point on the efficient frontier.

Customizing this same code to solve parametrically for every point on the efficient frontier, it took 23,446 pivots and 57 minutes to compute every point on the frontier.

The efficient frontier consists of 23,446 distinct portfolios. Click here for a partial list (warning: the file is 2.5 MBytes). The complete list makes a 37 MByte file.
Upside Risk—An Oxymoron

\[
\rho_2(x) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j D_{tj} x_j \right)^2 \Rightarrow \rho_2^-(x) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j D_{tj} x_j \right)^2,
\]

\[
\rho_1(x) = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j D_{tj} x_j \right| \Rightarrow \rho_1^-(x) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_j D_{tj} x_j \right).
\]

where \((x)_- = \min(x, 0)\).

**Theorem** (Trivial '02) \(\rho_1^-(x) = \rho_1(x)/2\).

**Corollary** Efficient frontier is “good” for all \(\lambda \geq 1\).

Note: No analogous result for semi-variance.
Means, Medians, and Quantiles

Mean is solution to: \( \min_z \sum_j (b_j - z)^2 \)

Median is solution to: \( \min_z \sum_j |b_j - z| \)

In MAD portfolio model, replace

\[
\left| \sum_j R_j(t)x_j - \mu(x) \right| \quad \text{with} \quad \left| \sum_j R_j(t)x_j - z \right|
\]

to get mean absolute deviation from the median.

**Theorem**   Efficient frontier is “good” for all \( \lambda \geq 1 \).
REVIEW

• A portfolio is *bad* if another portfolio dominates it (stochastically).

• Many portfolios on Markowitz’s “efficient frontier” are bad.

• MAD Markowitz isn’t bad.

• MAD Markowitz is a parametric LP.

• Even more, using the parametric simplex method the entire efficient frontier can be computed in the time normally required to find just one point on the frontier.

• Lastly, our efficient frontier is completely determined by a finite set of portfolios (vs. a continuum).

Paper was published a year ago in *Econometrica*. 
Michael Yang, CIO
Steven Kamara, Trade Associate

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