ORF 307: Lecture 6

Linear Programming: Chapter 5
Duality

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Revisiting a question from last lecture:

\[
\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!} = \frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdot \frac{2n-2}{n-2} \cdot \frac{2n-3}{n-3} \cdot \frac{2n-4}{n-4} \cdot \frac{2n-5}{n-5} \cdots \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{1} \cdot \frac{1}{1} \\
\leq \frac{2n}{n} \cdot \frac{2(n-1)}{n-1} \cdot \frac{2(n-2)}{n-2} \cdot \frac{2(n-3)}{n-3} \cdot \frac{2(n-4)}{n-4} \cdots \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{1} \cdot \frac{1}{1} \\
= 2^{2n} \\
= 4^n
\]

\[
\binom{n+m}{m} \leq \binom{n+m}{0} + \binom{n+m}{1} + \binom{n+m}{2} + \cdots + \binom{n+m}{n+m} \\
= (1 + 1)^{n+m} \\
= 2^{n+m}
\]
Given a cache of raw materials and a factory for turning these raw materials into a variety of finished products, how many of each product type should we make so as to maximize profit?

This is a resource allocation problem:

\[
\begin{align*}
\text{maximize} & \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to} & \quad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\
& \quad a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \\
& \quad \vdots \\
& \quad a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m \\
& \quad x_1, x_2, \ldots, x_n \geq 0,
\end{align*}
\]

where

- \( c_j \) = profit per unit of product \( j \) produced
- \( b_i \) = units of raw material \( i \) on hand
- \( a_{ij} \) = units raw material \( i \) required to produce one unit of prod \( j \).
If we produce one unit less of product $j$, then we free up $a_{ij}$ units of raw material $i$. Selling these unused raw materials for $y_1, y_2, \ldots, y_m$ dollars/unit yields

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \text{ dollars.}$$

Only interested if this exceeds lost profit on each product $j$:

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j, \quad j = 1, 2, \ldots, n.$$

Consider a buyer offering to purchase our entire inventory.

Subject to above constraints, buyer wants to minimize cost:

\[
\begin{align*}
\text{minimize} & \quad b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{subject to} & \quad a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
& \quad a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\
& \quad \vdots \quad \vdots \quad \vdots \\
& \quad a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\
& \quad y_1, y_2, \ldots, y_m \geq 0,
\end{align*}
\]
Duality

Every Problem:

\[
\text{maximize} \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, m \\
x_j \geq 0 \quad j = 1, 2, \ldots, n,
\]

Has a Dual:

\[
\text{minimize} \quad \sum_{i=1}^{m} b_i y_i \\
\text{subject to} \quad \sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1, 2, \ldots, n \\
y_i \geq 0 \quad i = 1, 2, \ldots, m.
\]
Dual of Dual

Primal Problem:

maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \]
\[ x_j \geq 0 \quad j = 1, \ldots, n \]

Original problem is called the primal problem.

A problem is defined by its data (notation used for the variables is arbitrary).

Dual in “Standard” Form:

−maximize \[ \sum_{i=1}^{m} -b_i y_i \]
subject to \[ \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j \quad j = 1, \ldots, n \]
\[ y_i \geq 0 \quad i = 1, \ldots, m \]

Dual is “negative transpose” of primal.

Theorem Dual of dual is primal.
Weak Duality Theorem

If \((x_1, x_2, \ldots, x_n)\) is feasible for the primal and \((y_1, y_2, \ldots, y_m)\) is feasible for the dual, then

\[
\sum_j c_j x_j \leq \sum_i b_i y_i.
\]

Proof.

\[
\sum_j c_j x_j \leq \sum_j \left( \sum_i y_i a_{ij} \right) x_j
= \sum_{ij} y_i a_{ij} x_j
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i
\leq \sum_i b_i y_i.
\]
Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?

Answer is provided by the Strong Duality Theorem (coming later).
Simplex Method and Duality

A Primal Problem:

\[
\text{maximize } \xi = 0 - 2x_1 + 1x_2 + 1x_3 + 4x_4 \\
w_1 = 4 - 0x_1 - 1x_2 - 1x_3 - 2x_4 \\
w_2 = 3 - 0x_1 - 0x_2 - 2x_3 - 2x_4 \\
w_3 = 5 - 6x_1 - 2x_2 - 2x_3 - 5x_4
\]

Its Dual:

\[
\text{maximize } -\xi = 0 - 4y_1 - 3y_2 - 5y_3 \\
z_1 = 2 - 0y_1 - 0y_2 - 6y_3 \\
z_2 = -1 - 1y_1 - 0y_2 - 2y_3 \\
z_3 = -1 - 1y_1 - 2y_2 + 2y_3 \\
z_4 = -4 - 2y_1 - 2y_2 - 5y_3
\]

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: \(x_4\) enters, \(w_3\) leaves.

Make analogous pivot in dual: \(z_4\) leaves, \(y_3\) enters.

Seed = 3, GenRand five times
Second Iteration

After First Pivot:

Primal (feasible):

\[
\begin{align*}
\text{maximize } \zeta &= 4 + \frac{34}{5} x_1 + \frac{13}{5} x_3 - 2\frac{12}{5} x_4 - 12\frac{2}{5} x_2 + \frac{9}{5} x_3 - 1\frac{12}{5} x_2 + \frac{4}{5} x_2 - 9\frac{9}{5} x_1 - 9\frac{9}{5} x_3 - 2\frac{2}{5} x_3 + \frac{1}{5} x_3 - 1\frac{2}{5} w_3 - 2\frac{2}{5} w_3 + \frac{1}{5} w_3 - 2\frac{1}{5} w_3 \\
\end{align*}
\]

Dual (still not feasible):

Note: negative transpose property intact.

Again, use primal to pick pivot: \( x_3 \) enters, \( w_2 \) leaves.

Make analogous pivot in dual: \( z_3 \) leaves, \( y_2 \) enters.
Third Iteration

After Second Pivot:

Primal (feasible):

\[
\begin{array}{c|cccc|c}
 & w_1 & x_1 & x_2 & x_3 & x_4 \\
\hline
 w_1 & 19/14 & -6/7 & 5/7 & -9/14 & -1/7 \\
 x_3 & 5/14 & -6/7 & -2/7 & 5/14 & -1/7 \\
 x_4 & 8/7 & 6/7 & 2/7 & 1/7 & 1/7 \\
\end{array}
\]

Dual (still not feasible):

\[
\begin{array}{c|cccc|c}
 & z_1 & y_1 & z_2 & z_3 & z_4 \\
\hline
 z_1 & 32/7 & 6/7 & 6/7 & 6/7 & -6/7 \\
 z_2 & -1/7 & -5/7 & 2/7 & 2/7 & -2/7 \\
 y_2 & 13/14 & 9/14 & -5/14 & -5/14 & -1/7 \\
 y_3 & 3/7 & 1/7 & 1/7 & 1/7 & -1/7 \\
\end{array}
\]

Note: *negative transpose property intact.*

Again, use primal to pick pivot: \( x_2 \) enters, \( w_1 \) leaves.

Make analogous pivot in dual: \( z_2 \) leaves, \( y_1 \) enters.
After Third Iteration

Primal:

- Is \textit{optimal}.

Dual:

- Negative transpose property remains intact.
- Is \textit{optimal}.

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.
A primal pivot:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$d$</td>
<td>$c$</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td></td>
</tr>
</tbody>
</table>

The corresponding dual pivot:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$-d$</td>
<td>$-b$</td>
<td></td>
</tr>
<tr>
<td>$-c$</td>
<td>$-a$</td>
<td></td>
</tr>
</tbody>
</table>

| $d - \frac{bc}{a}$ | $c/a$ |   |
| $-b/a$ | $1/a$ |   |

| $-d + \frac{bc}{a}$ | $b/a$ |   |
| $-c/a$ | $-1/a$ |   |
Conclusion on previous slide is the essence of the strong duality theorem which we now state:

**Theorem.** *If the primal problem has an optimal solution,*

\[ x^* = (x_1^*, x_2^*, \ldots, x_n^*), \]

*then the dual also has an optimal solution,*

\[ y^* = (y_1^*, y_2^*, \ldots, y_m^*), \]

*and*

\[ \sum_j c_j x_j^* = \sum_i b_i y_i^*. \]

**Paraphrase:**

If primal has an optimal solution, then there is no duality gap.
Duality Gap

Four possibilities:

- Primal optimal, dual optimal (no gap).
  \[ \iff \text{Strong Duality Theorem} \]
- Primal unbounded, dual infeasible (no gap).
  \[ \iff \text{Weak Duality Theorem} \]
- Primal infeasible, dual unbounded (no gap).
  \[ \iff \text{Weak Duality Theorem} \]
- Primal infeasible, dual infeasible (infinite gap).
  \[ \iff \text{See example below} \]

Example of infinite gap:

\[
\begin{align*}
\text{maximize} & \quad 2x_1 - x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad -x_1 + x_2 \leq -2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]