ORF 307: Lecture 3

Linear Programming: Chapter 13, Section 1
Portfolio Optimization

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Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences
in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR’S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS
AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize
in Economic Sciences with one third each, to

Professor Harry Markowitz, City University of New York, USA,
Professor Merton Miller, University of Chicago, USA,
Professor William Sharpe, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called,
Capital Asset Pricing Model (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources
among various areas of production. It is to a large extent through financial markets that saving in
different sectors of the economy is transferred to firms for investments in buildings and machines.
Financial markets also reflect firms’ expected prospects and risks, which implies that risks can be spread
and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry
Markowitz who developed a theory for households’ and firms’ allocation of financial assets under
uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally
invested in assets which differ in regard to their expected return and risk, and thereby also how risks can
be reduced.
Notation: $S_j(t) =$ share price for investment $j$ at time $t$. 

Historical Data—Some ETF Prices
Return Data: $R_j(t) = \frac{S_j(t)}{S_j(t - 1)}$

Important observation: volatility is easy to see, mean return is lost in the noise.
**Risk vs. Reward**

**Reward:** Estimated using historical means:

\[
\text{reward}_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t).
\]

**Risk:** Markowitz defined risk as the variability of the returns as measured by the historical variances:

\[
\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} (R_j(t) - \text{reward}_j)^2.
\]

However, to get a linear programming problem (and for other reasons) we use the sum of the absolute values instead of the sum of the squares:

\[
\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} |R_j(t) - \text{reward}_j|.
\]
Why Make a Portfolio? ... Hedging

Investment A: Up 20%, down 10%, equally likely—a risky asset.

Investment B: Up 20%, down 10%, equally likely—another risky asset.

Correlation: Up-years for A are down-years for B and vice versa.

Portfolio: Half in A, half in B: up 5% every year! No risk!
Explain the 5\% every year claim.
Note: Not much negative correlation in price fluctuations. An up-day is an up-day and a down-day is a down-day.
Portfolios

Fractions: \( x_j = \text{fraction of portfolio to invest in } j \)

Portfolio’s Historical Returns: \( R_x(t) = \sum_j x_j R_j(t) \)

Portfolio’s Reward: \[
\text{reward}(x) = \frac{1}{T} \sum_{t=1}^{T} R_x(t) = \frac{1}{T} \sum_{t=1}^{T} \sum_j x_j R_j(t) \\
= \sum_j x_j \frac{1}{T} \sum_{t=1}^{T} R_j(t) = \sum_j x_j \text{reward}_j
\]
What's a Good Formula for the Portfolio's Risk?

\[ \text{risk}(x) = ? \]
Portfolio’s Risk:

\[
\text{risk}(x) = \frac{1}{T} \sum_{t=1}^{T} \left| R_x(t) - \text{reward}(x) \right| \\
= \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j x_j R_j(t) - \frac{1}{T} \sum_{s=1}^{T} \sum_j x_j R_j(s) \right| \\
= \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j x_j \left( R_j(t) - \frac{1}{T} \sum_{s=1}^{T} R_j(s) \right) \right| \\
= \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j x_j (R_j(t) - \text{reward}_j) \right|
\]
A Markowitz-Type Model

**Decision Variables:** the fractions $x_j$.

**Objective:** maximize return, minimize risk.

**Fundamental Lesson:** can't simultaneously optimize two objectives.

**Compromise:** set an upper bound $\mu$ for risk and maximize reward subject to this bound constraint:

- Parameter $\mu$ is called **risk aversion parameter**.
- Large value for $\mu$ puts emphasis on reward maximization.
- Small value for $\mu$ puts emphasis on risk minimization.

**Constraints:**

$$\frac{1}{T} \sum_{t=1}^{T} \left| \sum_j x_j (R_j(t) - \text{reward}_j) \right| \leq \mu$$

$$\sum_j x_j = 1$$

$$x_j \geq 0 \quad \text{for all } j$$
Optimization Problem

maximize \[ \frac{1}{T} \sum_{t=1}^{T} \sum_{j} x_j R_j(t) \]

subject to \[ \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j} x_j (R_j(t) - \text{reward}_j) \right| \leq \mu \]

\[ \sum_{j} x_j = 1 \]

\[ x_j \geq 0 \quad \text{for all} \ j \]

Because of absolute values not a linear programming problem.

Easy to convert…
Using the “greedy substitution”, we introduce new variables to represent the troublesome part of the problem

\[ y_t = \left| \sum_j x_j (R_j(t) - \text{reward}_j) \right| \]

to get

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{T} \sum_{t=1}^{T} \sum_j x_j R_j(t) \\
\text{subject to} & \quad \left| \sum_j x_j (R_j(t) - \text{reward}_j) \right| = y_t \quad \text{for all } t \\
& \quad \frac{1}{T} \sum_{t=1}^{T} y_t \leq \mu \\
& \quad \sum_j x_j = 1 \\
& \quad x_j \geq 0 \quad \text{for all } j.
\end{align*}
\]

We then note that the constraint defining \( y_t \) can be relaxed to a pair of inequalities:

\[-y_t \leq \sum_j x_j (R_j(t) - \text{reward}_j) \leq y_t.\]
A Linear Programming Formulation

maximize \[ \frac{1}{T} \sum_{t=1}^{T} \sum_{j} x_j R_j(t) \]

subject to \[ -y_t \leq \sum_{j} x_j (R_j(t) - \text{reward}_j) \leq y_t \quad \text{for all } t \]

\[ \frac{1}{T} \sum_{t=1}^{T} y_t \leq \mu \]

\[ \sum_{j} x_j = 1 \]

\[ x_j \geq 0 \quad \text{for all } j \]

\[ y_t \geq 0 \quad \text{for all } t \]
set Assets;
set Dates;

param T := card(Dates);
param mu;
param returns {Dates,Assets};

param mean {j in Assets} := ( sum{t in Dates} returns[t,j] ) / T;
param returns_dev {t in Dates, j in Assets} := returns[t,j] - mean[j];
param meanAbsDev {j in Assets} := sum{t in Dates} abs(returns_dev[t,j]) / T;

var x{Assets} >= 0;
var y{Dates} >= 0;

maximize reward: sum{j in Assets} mean[j]*x[j] ;

s.t. risk_bound: sum{t in Dates} y[t] / T <= mu;
    s.t. tot_mass: sum{j in Assets} x[j] = 1;
    s.t. y_lo_bnd {t in Dates}: -y[t] <= sum{j in Assets} returns_dev[t,j]*x[j];
    s.t. y_up_bnd {t in Dates}: sum{j in Assets} returns_dev[t,j]*x[j] <= y[t];
set RiskReward := {'risk', 'reward'};
param numiters := 20;
param portfolio {0..numiters, Assets union RiskReward};
set assets_max_mean ordered := {j in Assets: mean[j] == max {jj in Assets} mean[jj]};
param maxrisk := meanAbsDev[first(assets_max_mean)];
param minrisk := min {j in Assets} meanAbsDev[j];

for {k in 0..numiters} {
    display k;
    let mu := (k/20)*minrisk + (1-k/20)*maxrisk;

    solve;

    let {j in Assets} portfolio[k,j] := x[j];
    let portfolio[k,'reward'] := reward;
    let portfolio[k,'risk'] := sum{t in Dates} abs(sum{j in Assets} returns_dev[t,j]*x[j]) / T;
}
Varying risk bound $\mu$ produces the so-called *efficient frontier*. Portfolios on the efficient frontier are reasonable. Portfolios not on the efficient frontier can be strictly improved.

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Efficient Frontier

![Efficient Frontier Chart]

- XLF
- XLV
- XLE
- XLP
- XLY
- XLU
- MDY
- XLU
- MDY
- QQQ
- XLI
- XLK
- SPX
- XLB

Risks vs. Mean Return (annualized)