Game Theory

John Nash = A Beautiful Mind
The other John Nash
Rock-Paper-Scissors

A two person game.

Rules.
At the count of three declare one of:

Rock     Paper     Scissors

Winner Selection. Identical selection is a draw. Otherwise:

• Rock dulls Scissors
• Paper covers Rock
• Scissors cuts Paper

Check out Sam Kass' version: Rock, Paper, Scissors, Lizard, Spock

It was featured on The Big Bang Theory.
Payoffs are from row player to column player:

\[
A = \begin{bmatrix}
R & P & S \\
\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}
\end{bmatrix}
\]

Note: Any deterministic strategy employed by either player can be defeated systematically by the other player.
Given: \( m \times n \) matrix \( A \).

- **Row player** selects a **strategy** \( i \in \{1, \ldots, m\} \).
- **Column player** selects a **strategy** \( j \in \{1, \ldots, n\} \).
- Row player pays column player \( a_{ij} \) dollars.

**Note:** The rows of \( A \) represent deterministic strategies for row player, while columns of \( A \) represent deterministic strategies for column player.

Deterministic strategies can be (and usually are) bad.
Randomized Strategies.

• Suppose row player picks \( i \) with probability \( y_i \).
• Suppose column player picks \( j \) with probability \( x_j \).

Throughout, \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \) and \( y = [y_1 \ y_2 \ \cdots \ y_m]^T \) will denote stochastic vectors:

\[
x_j \geq 0, \quad j = 1, 2, \ldots, n
\]
\[
\sum_j x_j = 1
\]

\[
y_i \geq 0, \quad i = 1, 2, \ldots, m
\]
\[
\sum_i y_i = 1
\]

If row player uses random strategy \( y \) and column player uses \( x \), then expected payoff from row player to column player is

\[
\sum_i \sum_j y_i a_{ij} x_j = y^T A x
\]
Suppose column player were to adopt strategy \( x \).

Then, row player’s best defense is to use strategy \( y \) that minimizes \( y^T Ax \):

\[
\min_y y^T Ax
\]

And so column player should choose that \( x \) which maximizes these possibilities:

\[
\max_x \min_y y^T Ax
\]
What’s the solution to this problem:

$$\text{minimize} \quad 3y_1 + 6y_2 + 2y_3 + 18y_4 + 7y_5$$

subject to:

$$y_1 + y_2 + y_3 + y_4 + y_5 = 1$$

$$y_i \geq 0, \quad i = 1, 2, 3, 4, 5$$
Inner optimization is easy:

$$\min_y y^T A x = \min_i e_i^T A x$$

($e_i$ denotes the vector that’s all zeros except for a one in the $i$-th position—that is, deterministic strategy $i$).

*Note:* Reduced a minimization over a *continuum* to one over a *finite set*.

We have:

$$\max (\min_i e_i^T A x)$$

$$\sum_j x_j = 1,$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n.$$
Introduce a scalar variable $v$ representing the value of the inner minimization:

\[
\begin{align*}
\text{max } v \\
v &\leq e_i^T A x, \quad i = 1, 2, \ldots, m, \\
\sum_j x_j &= 1, \\
x_j &\geq 0, \quad j = 1, 2, \ldots, n.
\end{align*}
\]

Writing in pure matrix-vector notation:

\[
\begin{align*}
\text{max } v \\
v e - A x &\leq 0 \\
e^T x &= 1 \\
x &\geq 0
\end{align*}
\]

($e$ without a subscript denotes the vector of all ones).
Finally, in Block Matrix Form

\[
\begin{align*}
\text{max} & \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ v \end{bmatrix} \\
\begin{bmatrix} -A & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} & \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
x & \geq 0 \\
v & \text{free}
\end{align*}
\]
Similarly, row player seeks $y^*$ attaining:

$$\min_y \max_x y^T A x$$

which is equivalent to:

$$\min u$$

$$ue - A^T y \geq 0$$
$$e^T y = 1$$
$$y \geq 0$$
Row Player’s Problem in Block-Matrix Form

\[
\min \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix}
\]

\[
\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[y \geq 0\]

\[u \text{ free}\]

**Note:** Row player’s problem is dual to column player’s:

\[
\max \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ v \end{bmatrix}
\]

\[
\begin{bmatrix} -A & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[x \geq 0\]

\[v \text{ free}\]

\[
\min \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix}
\]

\[
\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[y \geq 0\]

\[u \text{ free}\]
**MiniMax Theorem**

**Theorem.**

Let \( x^* \) denote column player’s solution to her max–min problem. Let \( y^* \) denote row player’s solution to his min–max problem. Then

\[
\max_x y^{*T}Ax = \min_y y^TAx^*.
\]

**Proof.** From **Strong Duality Theorem**, we have

\[
u^* = v^*.
\]

Also,

\[
v^* = \min_i e_i^TAx^* = \min_y y^TAx^*
\]

\[
u^* = \max_j y^{*T}Ae_j = \max_x y^{*T}Ax
\]

**QED**
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}:
    sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal:
    sum{j in COLS} x[j] = 1;
data;
set ROWS := P S R;
set COLS := P S R;
param A: P S R:=
    P  0  1 -2
    S -3  0  4
    R  5 -6  0
;
solve;
printf {j in COLS}: " %3s %10.7f \n", j, 102*x[j];
printf {i in ROWS}: " %3s %10.7f \n", i, 102*ineqs[i];
printf: "Value = %10.7f \n", 102*v;
ampl gamethy.mod
LOQO: optimal solution (12 iterations)
primal objective -0.1568627451
dual objective -0.1568627451
  P  40.00000000
  S  36.00000000
  R  26.00000000
  P  62.00000000
  S  27.00000000
  R  13.00000000
Value = -16.00000000
Consider:
\[
\max c^T x \\
Ax = b \\
x \geq 0
\]

Rewrite equality constraints as pairs of inequalities:
\[
\max c^T x \\
Ax \leq b \\
-Ax \leq -b \\
x \geq 0
\]

Put into block-matrix form:
\[
\max c^T x \\
\begin{bmatrix} A & -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix} \\
x \geq 0
\]

Dual is:
\[
\min \begin{bmatrix} b \\ -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \\
\begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c \\
y^+, y^- \geq 0
\]

Which is equivalent to:
\[
\min b^T (y^+ - y^-) \\
A^T (y^+ - y^-) \geq c \\
y^+, y^- \geq 0
\]

Finally, letting \( y = y^+ - y^- \), we get
\[
\min b^T y \\
A^T y \geq c \\
y \text{ free.}
\]
Summary

• Equality constraints $\implies$ free variables in dual.
• Inequality constraints $\implies$ nonnegative variables in dual.

Corollary:

• Free variables $\implies$ equality constraints in dual.
• Nonnegative variables $\implies$ inequality constraints in dual.
A Real-World Example

The Ultra-Conservative Investor

Consider again some historical investment data \((S_j(t))\):

As before, we can let \( R_{t,j} = S_j(t)/S_j(t - 1) \) and view \( R \) as a payoff matrix in a game between Fate and the Investor.
Fate’s Conspiracy

The columns represent pure strategies for our conservative investor. The rows represent how history might repeat itself. Of course, for tomorrow, Fate won’t just repeat a previous day’s outcome but, rather, will present some mixture of these previous days. Likewise, the investor won’t put all of her money into one asset. Instead she will put a certain fraction into each. Using this data in the game-theory AMPL model, we get the following mixed-strategy percentages for Fate and for the investor.

**Investor’s Optimal Asset Mix:**

- XLP 98.4
- XLU 1.6

**Mean Old Fate’s Mix:**

- 2011-08-08 55.9 \( \Leftrightarrow \) Black Monday (2011)
- 2011-08-10 44.1

The value of the game is the investor’s expected return, 96.2%, which is actually a loss of 3.8%.

The data can be downloaded from here: [http://finance.yahoo.com/q/hp?s=XLU](http://finance.yahoo.com/q/hp?s=XLU) Here, **XLU** is just one of the funds of interest.
Starting From 2012...

To Ignore Black Monday (2011)
### Investor’s Optimal Asset Mix:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>XLK</td>
<td>75.5</td>
</tr>
<tr>
<td>XLV</td>
<td>15.9</td>
</tr>
<tr>
<td>XLU</td>
<td>6.2</td>
</tr>
<tr>
<td>XLB</td>
<td>2.2</td>
</tr>
<tr>
<td>XLI</td>
<td>0.2</td>
</tr>
</tbody>
</table>

### Mean Old Fate’s Mix:

<table>
<thead>
<tr>
<th>Date</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2015-03-25</td>
<td>3.9</td>
</tr>
<tr>
<td>2014-04-10</td>
<td>1.7</td>
</tr>
<tr>
<td>2013-06-20</td>
<td>68.9</td>
</tr>
<tr>
<td>2012-11-07</td>
<td>13.9</td>
</tr>
<tr>
<td>2012-06-01</td>
<td>11.5</td>
</tr>
</tbody>
</table>

The value of the game is the investor’s expected return, 97.7%, which is actually a loss of 2.3%. 
Giving Fate Fewer Options

Thousands seemed unfair—How about 20...
Investor’s Optimal Asset Mix:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>MDY</td>
<td>83.7</td>
</tr>
<tr>
<td>XLE</td>
<td>13.2</td>
</tr>
<tr>
<td>XLF</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Mean Old Fate’s Mix:

<table>
<thead>
<tr>
<th>Date</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2015-03-25</td>
<td>11.5</td>
</tr>
<tr>
<td>2015-03-10</td>
<td>33.5</td>
</tr>
<tr>
<td>2015-03-06</td>
<td>55.0</td>
</tr>
</tbody>
</table>

The value of the game is the investor’s expected return, 98.7%, which is actually a loss of 1.3%.