Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives

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February 3, 2011

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## Contents

1 The Black–Scholes Theory of Derivative Pricing .................................................. 11
  1.1 Market Model ..................................................................................................... 11
    1.1.1 Brownian Motion ...................................................................................... 11
    1.1.2 Stochastic Integrals .................................................................................. 12
    1.1.3 Risky Asset Price Model .......................................................................... 13
    1.1.4 Itô’s Formula ........................................................................................... 15
    1.1.5 Lognormal Risky Asset Price ................................................................... 16
    1.1.6 Ornstein–Uhlenbeck Process ................................................................... 17
  1.2 Derivative Contracts .......................................................................................... 18
    1.2.1 European Call and Put Options ............................................................... 18
    1.2.2 American Options .................................................................................... 19
    1.2.3 Other Exotic Options .............................................................................. 19
  1.3 Replicating Strategies ....................................................................................... 20
    1.3.1 Replicating Self-Financing Portfolios .................................................... 20
    1.3.2 The Black–Scholes Partial Differential Equation .................................... 21
    1.3.3 Pricing to Hedge ..................................................................................... 22
    1.3.4 The Black–Scholes Formula .................................................................... 23
    1.3.5 The Greeks .............................................................................................. 25
  1.4 Risk-Neutral Pricing .......................................................................................... 26
    1.4.1 Equivalent Martingale Measure ............................................................... 26
    1.4.2 Self-Financing Portfolios ........................................................................ 28
    1.4.3 Risk-Neutral Valuation .......................................................................... 28
    1.4.4 Using the Markov Property ..................................................................... 29
  1.5 Risk-Neutral Expectations and Partial Differential Equations ...................... 30
    1.5.1 Infinitesimal Generators and Associated Martingales ......................... 31
    1.5.2 Conditional Expectations and Parabolic Partial Differential Equations .... 32
    1.5.3 Kolmogorov Equations ........................................................................... 33
    1.5.4 Application to the Black–Scholes Partial Differential Equation ............ 34
  1.6 American Options and Free-boundary Problems ............................................. 35
    1.6.1 Optimal Stopping .................................................................................... 35
    1.6.2 Free-Boundary Value Problems ............................................................. 36
  1.7 Path-Dependent Derivatives ............................................................................. 37
    1.7.1 Barrier Options ....................................................................................... 37
    1.7.2 Lookback Options ................................................................................... 40
    1.7.3 Forward-Start Options .......................................................................... 41
    1.7.4 Compound Options ................................................................................ 42
    1.7.5 Asian Options ......................................................................................... 43
1.8 First Passage Structural Approach to Default .......................... 43
  1.8.1 Merton’s Approach .................................................. 43
  1.8.2 First Passage Model ................................................ 44
1.9 Multidimensional Stochastic Calculus .................................. 45
  1.9.1 Multi-dimensional Itô’s Formula ................................. 46
  1.9.2 Girsanov’s Theorem ................................................ 46
  1.9.3 The Feynman–Kac Formula ...................................... 47
1.10 Complete Market ....................................................... 48
1.11 Notes ................................................................. 48

2 Introduction to Stochastic Volatility Models ............................. 49
  2.1 Implied Volatility Surface ............................................. 49
    2.1.1 Interpretation of the Skew ...................................... 50
    2.1.2 What Data to Use ................................................. 52
  2.2 Local Volatility ....................................................... 54
    2.2.1 Time-dependent Volatility ..................................... 54
    2.2.2 Level-dependent Volatility and Dupire’s Formula .......... 55
  2.3 Stochastic Volatility Models ...................................... 57
    2.3.1 One-Factor Stochastic Volatility Models .................... 57
    2.3.2 Stock Price Distribution under Stochastic Volatility .... 59
  2.4 Derivative Pricing .................................................... 59
    2.4.1 Hedging Argument .............................................. 60
    2.4.2 Pricing with Equivalent Martingale Measures ............. 62
    2.4.3 Market Price of Volatility Risk and Data .................. 64
    2.4.4 Short-time tight fit vs. long-time approximate fit: the \( (K,T,t) \)-problem ................... 65
  2.5 General Results on Stochastic Volatility Models .................. 66
    2.5.1 Implied Volatility as a Function of Moneyness .......... 66
    2.5.2 Uncorrelated Volatility, Hull–White Formula & Renault–Touzi Theorem 67
    2.5.3 Correlated Stochastic Volatility ............................. 69
    2.5.4 Multi-factor Stochastic Volatility Models .................... 70
  2.6 Summary & Conclusions .............................................. 72
  2.7 Notes ................................................................. 73

3 Volatility Time Scales .................................................... 75
  3.1 A Simple Picture of Fast and Slow Time Scales .................... 75
  3.2 Ergodicity & Mean-reversion ..................................... 76
  3.3 Examples of Mean-Reverting Processes .............................. 82
    3.3.1 Example: Markov Chain ...................................... 82
    3.3.2 Example: Another Jump Process .............................. 84
    3.3.3 Example: The Ornstein–Uhlenbeck Process ................ 86
    3.3.4 Example: The Cox–Ingersoll–Ross (CIR) Process .......... 88
    3.3.5 Other One-Dimensional Diffusion Processes ............... 91
  3.4 Time-scales in Synthetic Returns data .............................. 94
    3.4.1 The Returns Time Series ..................................... 94
    3.4.2 Returns Process with Jump Volatility ...................... 95
    3.4.3 Returns Process with OU Volatility ....................... 95
  3.5 Time-Scales in Market Data ....................................... 96
    3.5.1 Short-term Volatility Statistics of the S&P 500 ........... 97
6.2.2 Explicit Expressions .............................................. 153
6.3 Asian Options .................................................. 154
6.4 Notes .......................................................... 156

7 Application to American Derivatives ............................................. 157
  7.1 American Problem under Stochastic Volatility .................. 157
  7.2 Stochastic Volatility Correction for American Put ............. 159
  7.3 Parameter Reduction ........................................... 162
  7.4 Summary ...................................................... 163
  7.5 Notes ........................................................ 163

8 Hedging Strategies .................................................. 165
  8.1 Black–Scholes Delta Hedging .................................. 165
  8.2 The Strategy and its Cost ..................................... 166
    8.2.1 Averaging Effect ....................................... 167
    8.2.2 Small Noise Effect and Summary of the Cost .......... 169
  8.3 Mean Self-Financing Hedging Strategy ......................... 170
  8.4 A Strategy with Frozen Parameters .......................... 172
    8.4.1 The Black–Scholes Strategy with Frozen Volatility ...... 172
    8.4.2 The Small Noise Effect .................................. 174
    8.4.3 Bias Reduction .......................................... 175
  8.5 Strategies Based on Implied Volatilities ..................... 178
    8.5.1 Strategy Based on Dynamically Calibrated Parameters ...... 178
    8.5.2 Staying Close to the Price ................................ 179
  8.6 Martingale Approach to Pricing ................................ 180
    8.6.1 Main Argument ......................................... 180
    8.6.2 Decomposition Result .................................. 181
    8.6.3 Comparison with the PDE Approach ..................... 184
  8.7 Non Markovian Models of Volatility ........................ 185
    8.7.1 Setting: an Example .................................... 185
    8.7.2 Asymptotics in the Non Markovian Case ................ 186
  8.8 Notes ........................................................ 188

9 Extensions .......................................................... 189
  9.1 Dividends and Varying Interest Rates ......................... 189
    9.1.1 Dividends ............................................... 189
    9.1.2 Varying Interest Rates .................................. 190
  9.2 Probabilistic Representation of the Approximated Prices .... 193
  9.3 Second Order Correction from Fast Scale .................... 194
    9.3.1 Expansion and Successive Equations ................... 194
    9.3.2 Computation of the Second Correction .................. 196
    9.3.3 Accuracy and Parameter Reduction ..................... 197
    9.3.4 Correction to the Skew .................................. 199
  9.4 Second Order Corrections from Slow and Fast Scales ........ 200
  9.5 Periodic Day Effect .......................................... 202
  9.6 Markovian Jump Volatility Models .......................... 204
  9.7 Multi-Dimensional Models .................................... 205
  9.8 Notes ........................................................ 209
10 Around the Heston Model

10.1 The Heston Model

10.1.1 European Derivatives Under The Heston Model
10.1.2 Numerical Evaluation of Call Options

10.2 Approximations to the Heston Model

10.2.1 Fast Mean-reverting Heston’s Model
10.2.2 Second-order Approximation for a Fast Mean-reverting Heston Model
10.2.3 Slowly varying Heston’s Model

10.3 A Fast Mean-Reverting Correction to the Heston Model

10.3.1 A Multiscale Generalized Heston Model
10.3.2 Pricing Equation
10.3.3 Asymptotic Analysis
10.3.4 The Multiscale Implied Volatility Surface

10.4 Large Deviations and Short Maturity Asymptotics

10.4.1 Fast Mean-Reversion and Short Maturity Scaling
10.4.2 Moment Generating Function and its Asymptotic
10.4.3 Applications to Pricing and Implied Volatilities

10.5 Notes

11 Other Applications

11.1 Application to Variance Reduction in Monte Carlo Computations

11.1.1 Importance Sampling
11.1.2 Control Variates

11.2 Portfolio Optimization under Stochastic Volatility

11.2.1 Constant Volatility Merton Problem
11.2.2 Stochastic Volatility Merton Problem
11.2.3 A Practical Solution

11.3 Application to CAPM Forward-looking Beta Estimation

11.3.1 Discrete-time CAPM and Forward-looking Beta’s
11.3.2 Continuous-time CAPM with Stochastic Volatility
11.3.3 Pricing Risk-Neutral Measure
11.3.4 Market and Asset Option Prices
11.3.5 Beta Estimation
11.3.6 Examples

11.4 Notes

12 Interest Rate Models

12.1 The Vasicek Model

12.1.1 Bond Pricing in the Vasicek Model
12.1.2 Yield Curve
12.1.3 Bond Volatility
12.1.4 Forward Rates

12.2 The Bond Price and its Expansion

12.2.1 The Full Bond-Pricing Problem
12.2.2 Bond Price Expansion
12.2.3 The Fast-Scale Correction
12.2.4 The Slow-Scale Correction
12.2.5 Combined Corrections
Introduction

This book is about pricing and hedging financial derivatives under stochastic volatility in equity, interest-rate and credit markets. We demonstrate that the introduction of two time scales in volatility, a fast and a slow, is needed and efficient for capturing the main features of the observed term structures of implied volatility, yields or credit spreads. The present book builds on and replaces our previous book “Derivatives in Financial Markets with Stochastic Volatility” published by Cambridge University Press in 2000.

We present an approach to derivatives valuation and hedging which consists of integrating singular and regular perturbation techniques in the context of stochastic volatility. The book has a dual purpose: to present “off-the-shelf” formulas and calibration tools, and to introduce, explain and develop the mathematical framework to handle the multiscale asymptotics.

There are many books on Financial Mathematics (mostly for introductory courses at the level of the Black-Scholes model). Primarily, these books deal with the case of constant volatilities, be it for stock prices, interest rates or default intensities. This book is about analyzing these models in the presence of stochastic volatility using the powerful tools of perturbation methods. The book can be used for a second level graduate course in Financial and Applied Mathematics.

Our goal is to address the following fundamental problem in pricing and hedging derivatives: how can traded call and put options, quoted in terms of implied volatilities, be used to price and hedge more complicated contracts? Modeling the underlying asset usually involves the specification of a multi-factor Markovian model under the risk-neutral pricing measure. Calibration of the parameters of that model to the observed implied volatilities, including the market prices of risk, is a challenging task because of the complex relation between option prices and model parameters (through a pricing partial differential equation, for instance). The main difficulty is to find models which will produce stable parameter estimates. We like to think of this problem as the “(K, T, t)-problem”: for a given present time t and a fixed maturity T, it is usually easy with low dimensional models to fit the skew with respect to strikes K. Getting a good fit of the term structure of implied volatility, that is when a range of observed maturities T is taken into account, is a much harder problem which can be handled with a sufficient number of parameters. The main problem remains: the stability with respect to t of these calibrated parameters. This is a crucial quality to have if one wants to use the model to compute no-arbitrage prices of more complex path-dependent derivatives, since in this case the distribution over time of the underlying is central.

Modeling directly the evolution of the implied volatility surface is a promising approach but involves some complicated issues. One has to make sure that the model is free of arbitrage or, in other words, that the surface is produced by some underlying under a risk-neutral measure. This is known to be a difficult task, and the choice of a model and its calibration is also an important issue in this approach. But most importantly, in order to use this modeling to price other path-dependent contracts, one has to identify a corresponding underlying which typically does not lead to a low dimensional Markovian evolution.

Wouldn’t it be nice to have a direct and simple connection between the observed implied
volatilities and prices of more complex path-dependent contracts! Our objective is to provide such a linkage. This is done by using a combination of singular and regular perturbations techniques corresponding respectively to fast and slow time scales in volatility. We obtain a parametrization of the implied volatility surface in terms of Greeks, which involves four parameters at the first order of approximation. This procedure leads to parameters which are exactly those needed to price other contracts at this level of approximation. In our previous work presented in (Fouque, Papanicolaou and Sircar 2000) we used only the fast volatility time scale combined with a statistical estimation of an effective constant volatility from historical data. The introduction of the slow volatility time scale enables us to capture more accurately the behavior of the term structure of implied volatility at long maturities. Yet, we preserve a parsimonious parametrization which effectively and robustly captures the main effects of time-scale heterogeneity. Moreover, in the framework presented here, statistics of historical data are not needed for the calibration of these parameters.

Thus, in summary, we directly link the implied volatilities to prices of path-dependent contracts by exploiting volatility time scales. Furthermore, we extend this approach to interest-rate and credit derivatives.

In Chapter 1 we review the basic ideas and methods of the Black-Scholes theory as well as the tools of stochastic calculus underpinning the models used. Chapter 2 provides a general introduction to stochastic volatility models. In Chapter 3, the key modeling chapter, we identify time scales in financial data and we introduce them in stochastic volatility models. In Chapter 4 we present the first-order perturbation theory in the context of European equity derivatives and identify the important parameters arising in this asymptotic analysis. This is the central chapter on the mathematical tools used in our multiscale modeling approach. In Chapter 5 we provide a calibration procedure for these parameters using observed implied volatilities. Indeed, these are the parameters that provide a parsimonious linkage between various contracts. We also show in this chapter how to extend the perturbation techniques to the case with time-dependent parameters needed for practical fitting of the presented S&P 500 data. The extensions to exotic and American claims are described in Chapters 6 and 7. It is also natural to exploit the presence of a skew of implied volatilities for designing hedging strategies of part of the volatility risk by trading the underlying. This is achieved in Chapter 8 by using the asymptotic analysis presented in the previous chapters combined with a martingale argument which in turns can be used to derive asymptotics in the case of non Markovian models of volatility. In Chapter 9 we present several extensions to the perturbation theory including the cases with dividends and varying interest rates, and the derivation of the second order corrections. Next, in Chapter 10, we discuss the Heston model which is very popular for its computational tractability. We implement our perturbation theory on this particular model, we show how to generalize it while retaining its tractability, and we derive large deviation results in the regime of short maturities and fast mean-reverting volatility. Applications to variance reduction techniques for Monte Carlo simulations, to portfolio optimization, and to estimation of CAPM Beta parameters are presented in Chapter 11. After introducing the basics of fixed-income markets, we demonstrate in Chapter 12 that our perturbation approach is also effective for interest-rates models with stochastic volatility. Then, we introduce the fundamental concepts used in credit risk modeling, and we apply our method to both single-name and multi-name credit derivatives using structural models in Chapter 13 and intensity-based models in Chapter 14.

One cannot write a book in 2010 on financial mathematics without commenting on the recent financial crisis. We choose to do so in the Epilogue–Chapter 15 since it involves judgement and behavior of the market players rather than mathematical modeling as presented in this book.
Chapter 1

The Black–Scholes Theory of Derivative Pricing

The aim of this first chapter is to review the basic objects, ideas, and results of the classical Black–Scholes theory of derivative pricing. It is intended for readers who want to enter the subject or simply refresh their memory. This is not a complete treatment of this theory with detailed proofs but rather an intuitive but precise presentation including a few key calculations. Detailed presentations of the subject can be found in many books at various levels of mathematical rigor and generality, a few of which we list in the notes at the end of the chapter.

This book is about extending the Black–Scholes theory using perturbation methods in order to handle markets with stochastic volatility. The notation and many of the tools used in the constant volatility case will be used for the more complex markets throughout the book.

1.1 Market Model

In this simple model, suggested by Samuelson and used by Black and Scholes, there are two assets. One is a riskless asset (bond) with price \( \beta_t \) at time \( t \) described by the ordinary differential equation

\[
d\beta_t = r\beta_t dt,
\]

(1.1)

where \( r \), a non-negative constant, is the instantaneous interest rate for lending or borrowing money. Setting \( \beta_0 = 1 \), we have \( \beta_t = e^{rt} \) for \( t \geq 0 \). The price \( X_t \) of the other asset, the risky stock or stock index, evolves according to the stochastic differential equation

\[
dX_t = \mu X_t dt + \sigma X_t dW_t,
\]

(1.2)

where \( \mu \) is a constant mean return rate, \( \sigma > 0 \) is a constant volatility and \((W_t)_{t \geq 0}\) is a standard Brownian motion. This fundamental model and the intuitive content of equation (1.2) are presented in the following sections.

1.1.1 Brownian Motion

The Brownian motion is a stochastic process whose definition, existence, properties, and applications have been (and still are) the subject of numerous studies during the twentieth century. Our goal here is to give a very intuitive and practical presentation.

A Brownian motion is a real-valued stochastic process with continuous trajectories that have independent and stationary increments. The trajectories are denoted by \( t \to W_t \) and for the standard Brownian motion, we have that:
• $W_0 = 0$;
• for any $0 < t_1 < \cdots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$ are independent;
• for any $0 \leq s < t$, the increment $W_t - W_s$ is a centered (mean-zero) normal random variable with variance $IE\{(W_t - W_s)^2\} = t - s$. In particular $W_t$ is $N(0,t)$-distributed.

We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space where our Brownian motion is defined and the expectation $IE\{\cdot\}$ is computed. For instance, it could be $\Omega = \mathcal{C}([0, +\infty) : \mathbb{R})$, the space of all continuous trajectories $\omega$, with $W_t(\omega) = \omega(t)$. The $\sigma$-algebra $\mathcal{F}$ contains all sets of the form $\{\omega \in \Omega : |\omega(s)| < R, s \leq t\}$; $\mathbb{P}$ is the Wiener measure, which is the probability distribution of the standard Brownian motion.

The increasing family of $\sigma$-algebras $\mathcal{F}_t$ generated by $(W_s)_{s \leq t}$, the information on $W$ up to time $t$, and all the sets of probability 0 in $\mathcal{F}$, is called the natural filtration of the Brownian motion. This completion by the null sets is important, in particular for the following reason. If two random variables $X$ and $Y$ are equal almost surely $(X = Y \mathbb{P}$-a.s. means $\mathbb{P}\{X = Y\} = 1)$ and if $X$ is $\mathcal{F}_t$-measurable (meaning that any event $\{X_t \leq x\}$ belongs to $\mathcal{F}_t$) then $Y$ is also $\mathcal{F}_t$-measurable.

A stochastic process $(X_t)_{t \geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ if the random variable $X_t$ is $\mathcal{F}_t$-measurable for every $t$. We say that $(X_t)$ is $(\mathcal{F}_t)$-adapted. If another process $(Y_t)$ is such that $X_t = Y_t$ $\mathbb{P}$-a.s. for every $t$ then it is also $(\mathcal{F}_t)$-adapted.

The independence of the increments of the Brownian motion and their normal distribution can be summarized using conditional characteristic functions. For $0 \leq s < t$ and $u \in \mathbb{R}$

$$IE\left\{e^{iu(W_t-W_s)} \mid \mathcal{F}_s\right\} = e^{-\frac{u^2(t-s)}{2}},$$

(1.3)

If $W$ is a Brownian motion, by independence of the increment $W_t - W_s$ from the past $\mathcal{F}_s$, the left-hand side of (1.3) is simply $IE\{e^{iu(W_t-W_s)}\}$, which is the characteristic function of a centered normal random variable with variance $t-s$, and is equal to the right-hand side. Conversely, if (1.3) holds, then the continuous process $(W_t)$ is a standard Brownian motion.

This independence of increments makes the Brownian motion an ideal candidate for defining a complete family of independent infinitesimal increments $dW_t$, which are centered, normally distributed with variance $dt$ and which will serve as a model of (Gaussian white) noise. The drawback is that the trajectories of $(W_t)$ cannot be “nice” in the sense that they are not of bounded variation, as the following simple computation suggests. Let $t_0 = 0 < t_1 < \cdots < t_n = t$ be a subdivision of $[0, t]$, which we may suppose evenly spaced so that $t_i - t_{i-1} = t/n$ for each interval. The quantity

$$IE\left\{\sum_{i=1}^n |W_{t_i} - W_{t_{i-1}}|\right\} = nIE\{|W_\frac{t}{n}|\} = n\sqrt{\frac{t}{n}}IE\{|W_1|\},$$

goins to $+\infty$ as $n \to +\infty$, indicating that the integral with respect to $dW_t$ cannot be defined in the usual way “trajectory by trajectory”. We describe how such integrals can be defined in the next section.

1.1.2 Stochastic Integrals

For $T$ a fixed finite time, let $(X_t)_{0 \leq t \leq T}$ be a stochastic process adapted to $(\mathcal{F}_t)_{0 \leq t \leq T}$, the filtration of the Brownian motion up to time $T$, such that

$$IE\left\{\int_0^T X_t^2 dt\right\} < +\infty.$$
Using iterated conditional expectations and the independent increments property of Brownian motion, we note that

$$IE \left\{ \left( \sum_{i=1}^{n} X_{t_{i-1}} (W_{t_{i}} - W_{t_{i-1}}) \right)^2 \right\} = IE \left\{ \sum_{i=1}^{n} (X_{t_{i-1}})^2 (t_{i} - t_{i-1}) \right\},$$

for $t \leq T$, which is a basic calculation in the construction of stochastic integrals. Note also that the Brownian increments on the left are forward in time and that the sum on the right converges to $IE \left\{ \int_{0}^{t} X_{s}^2 ds \right\}$ which is finite by (1.4).

The stochastic integral of $(X_t)$ with respect to the Brownian motion $(W_t)$ is defined as a limit in the mean-square sense ($L^2(\Omega)$)

$$\int_{0}^{t} X_{s}dW_{s} = \lim_{n \rightarrow +\infty} \sum_{i=1}^{n} X_{t_{i-1}} (W_{t_{i}} - W_{t_{i-1}}), \quad (1.5)$$
as the mesh size of the subdivision goes to zero.

As a function of time $t$, this stochastic integral defines a continuous square integrable process such that

$$IE \left\{ \left( \int_{0}^{t} X_{s}dW_{s} \right)^2 \right\} = IE \left\{ \int_{0}^{t} X_{s}^2 ds \right\}, \quad (1.6)$$

and has the martingale property

$$IE \left\{ \int_{0}^{t} X_{u}dW_{u} \mid \mathcal{F}_{s} \right\} = \int_{0}^{s} X_{u}dW_{u} \text{ P-a.s., for } s \leq t, \quad (1.7)$$
as can be easily deduced from the definition (1.5). The quadratic variation $(Y)_t$ of the stochastic integral $Y_t = \int_{0}^{t} X_{u}dW_{u}$ is

$$(Y)_t = \lim_{n \rightarrow +\infty} \sum_{i=1}^{n} (Y_{t_{i}} - Y_{t_{i-1}})^2 = \int_{0}^{t} X_{s}^2 ds \quad (1.8)$$
in the mean-square sense.

Stochastic integrals are zero mean, continuous and square integrable martingales. It is interesting to note that the converse is also true: every zero mean, continuous and square integrable martingale is a Brownian stochastic integral. This representation result will be made precise and used in Section 1.4.

1.1.3 Risky Asset Price Model

The Black–Scholes model for the risky asset price corresponds to a continuous process $(X_t)$ such that, in an infinitesimal amount of time $dt$, the infinitesimal return $dX_t/X_t$ has mean $\mu dt$, proportional to $dt$, with a constant rate of return $\mu$, and centered random fluctuations independent of the past up to time $t$. These fluctuations are modeled by $\sigma dW_t$ where $\sigma$ is a positive constant volatility which measures the strength of the noise, and $dW_t$ the infinitesimal increments of the Brownian motion. The corresponding formula for the infinitesimal return is

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t, \quad (1.9)$$
which is the stochastic differential equation (1.2). The right-side has the natural financial interpretation of a return term plus a risk term. We are also assuming that there are no dividends paid in the time interval that we are considering. It is easy to incorporate a continuous dividend rate in all that follows, but for simplicity we shall omit this here.

In integral form, this equation is

\[ X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s, \tag{1.10} \]

where the last integral is a stochastic integral as described in Section 1.1.2 and where \( X_0 \) is the initial value, which is assumed to be independent of the Brownian motion and square integrable.

Equation (1.10), or (1.2) in the differential form, is a particular case of a general class of stochastic differential equations driven by a Brownian motion:

\[ dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \tag{1.11} \]

or in integral form

\[ X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s. \tag{1.12} \]

In the Black–Scholes model, \( \mu(t, x) = \mu x \) and \( \sigma(t, x) = \sigma x \); these are independent of \( t \), differentiable in \( x \), and linearly growing at infinity (since they are linear). This is enough to ensure existence and uniqueness of a continuous adapted and square integrable solution \((X_t)\). The proof of this result is based on simple estimates like

\[ \mathbb{E} \{ X_t^2 \} = \mathbb{E} \left\{ \left( X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s \right)^2 \right\} \leq 3 \left( \mathbb{E} \{ X_0^2 \} + (\mu^2 T + \sigma^2) \int_0^t \mathbb{E} \{ X_s^2 \} ds \right), \]

where we have used the inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\), the Cauchy–Schwarz inequality

\[ \mathbb{E} \left( \int_0^t X_s ds \right)^2 \leq t \int_0^t \mathbb{E} \{ X_s^2 \} ds, \]

and (1.6). We deduce

\[ 0 \leq \mathbb{E} \{ X_t^2 \} \leq c_1 + c_2 \int_0^t \mathbb{E} \{ X_s^2 \} ds, \]

for \( 0 \leq t \leq T \) and constants \( c_1 \) and \( c_2 \geq 0 \). By a direct application of Gronwall’s lemma, we deduce that the solution is a priori square integrable. The construction of a solution and the proof of uniqueness can be obtained by similar and slightly more complicated estimates that use the Kolmogorov-Doob inequality for martingales.

Looking at equation (1.9), it is very tempting to write \( X_t/X_0 \) explicitly as the exponential of \((\mu t + \sigma W_t)\). However, this is not correct because the usual chain rule is not valid for stochastic differentials. For instance \( W_t^2 \) is not equal to \( 2 \int_0^t W_s dW_s \) as might be expected since, by the martingale property (1.7), this last integral has an expectation equal to zero but \( \mathbb{E} \{ W_t^2 \} = t \).

This discrepancy is corrected by Itô’s formula explained in the following section.
1.1.4 Itô’s Formula

A function of the Brownian motion $W_t$ defines a new stochastic process $g(W_t)$. We suppose in the following that the function $g$ is twice continuously differentiable, bounded, and has bounded derivatives. The purpose of the chain rule is to compute the differential $dg(W_t)$ or equivalently its integral $g(W_t) - g(W_0)$. Using the subdivision $t_0 = 0 < t_1 \cdots < t_n = t$ of $[0, t]$, we write

$$g(W_t) - g(W_0) = \sum_{i=1}^{n} (g(W_{t_i}) - g(W_{t_{i-1}})).$$

We then apply Taylor’s formula to each term to obtain

$$g(W_t) - g(W_0) = \sum_{i=1}^{n} g'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})$$

$$+ \frac{1}{2} \sum_{i=1}^{n} g''(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2 + R,$$

where $R$ contains all the higher order terms.

If $(W_t)$ were differentiable only the first sum would contribute to the limit as the mesh size of the subdivision goes to zero, leading to the chain rule $dg(W_t) = g'(W_t)W_t dt$ of classical calculus. In the Brownian case, $(W_t)$ is not differentiable and, by (1.5), the first sum converges to the stochastic integral

$$\int_0^t g'(W_s)dW_s.$$

The correction comes from the second sum which, like (1.8), converges to

$$\frac{1}{2} \int_0^t g''(W_s)ds,$$

as can be seen by comparing it in $L^2$ with $\frac{1}{2} \sum_{i=1}^{n} g''(W_{t_{i-1}})(t_i - t_{i-1})$. The higher order terms contained in $R$ converge to zero and do not contribute to the limit, which is

$$g(W_t) - g(W_0) = \int_0^t g'(W_s)dW_s + \frac{1}{2} \int_0^t g''(W_s)ds. \quad (1.13)$$

This is the simplest version of Itô’s formula. It is often written in differential form:

$$dg(W_t) = g'(W_t)dW_t + \frac{1}{2}g''(W_t)dt. \quad (1.14)$$

The next step is deriving a similar formula for $dg(X_t)$ where $X_t$ is the solution of a stochastic differential equation like (1.11). We give here this general formula for a function $g$ depending also on time $t$:

$$dg(t, X_t) = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t)d\langle X \rangle_t, \quad (1.15)$$

where $dX_t$ is given by the stochastic differential equation (1.11) and

$$\langle X \rangle_t = \int_0^t \sigma^2(s, X_s)ds.$$
is the quadratic variation of the martingale part of $X_t$, that is, the stochastic integral on the right side of (1.12). In terms of $dt$ and $dW_t$ the formula is

$$dg(t, X_t) = \left( \frac{\partial g}{\partial t} + \mu(t, X_t) \frac{\partial g}{\partial x} + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2 g}{\partial x^2} \right) dt + \sigma(t, X_t) \frac{\partial g}{\partial x} dW_t,$$

where all the partial derivatives of $g$ are evaluated at $(t, X_t)$.

As an application we can compute the differential of the discounted price $g(t, X_t) = e^{-rt}X_t$:

$$d\left(e^{-rt}X_t\right) = -re^{-rt}X_t dt + e^{-rt}dX_t$$

$$= e^{-rt} \left(-rX_t + \mu(t, X_t)\right) dt + e^{-rt}\sigma(t, X_t)dW_t,$$  

(1.17)

since the second derivative of $g(t, x) = xe^{-rt}$ with respect to $x$ is zero. In the particular case of the price $X_t$ given by (1.2), $\mu(t, x) = \mu x$ and $\sigma(t, x) = \sigma x$ so we obtain

$$d\left(e^{-rt}X_t\right) = (\mu - r)\left(e^{-rt}X_t\right) dt + \sigma \left(e^{-rt}X_t\right) dW_t.$$  

(1.18)

The discounted price $\tilde{X}_t = e^{-rt}X_t$ satisfies the same equation as $X_t$ where the return $\mu$ has been replaced by $\mu - r$.

### 1.1.5 Lognormal Risky Asset Price

Coming back to the stochastic differential equation (1.9) for the evolution of the stock price $X_t$, it is natural to suspect from the ordinary calculus formula $\int dx/x = \log x$ that $\log X_t$ might satisfy an equation that we can integrate explicitly. We compute the differential of $\log X_t$ by applying Itô’s formula (1.16) with $g(t, x) = \log x$, $\mu(t, x) = \mu x$ and $\sigma(t, x) = \sigma x$:

$$d \log X_t = \left( \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t.$$

The logarithm of the stock price is then given explicitly by

$$\log X_t = \log X_0 + \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t,$$

which leads to the following formula for the stock price:

$$X_t = X_0 \exp \left( (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right),$$

(1.19)

The return $X_t/X_0$ is lognormal: it is the exponential of a nonstandard Brownian motion which is normally distributed with mean $(\mu - \frac{1}{2}\sigma^2) t$ and variance $\sigma^2 t$ at time $t$. The process $(X_t)$ is also called geometric Brownian motion. The stock price given by (1.19) satisfies equation (1.9). It can also be obtained as a diffusion limit of binomial tree models which arise when Brownian motion is approximated by a random walk.

Notice that, if $X_0 = 0$, $X_t$ stays at zero at all times thereafter. Thus in this model, bankruptcy (zero stock price) is a permanent state. However, $W_t$ is finite at all times, and therefore, if $X_0 > 0$, $X_t$ remains positive at all times.

In Figure 1.1, we show a sample path or realization of a geometric Brownian motion $(X_t)$ in which $\mu = 0.15$, $\sigma = 0.1$ and $X_0 = 95$. This path exhibits the “average growth plus noise” behaviour we expect from this model of asset prices.
Figure 1.1: A sample path of a geometric Brownian motion defined by the stochastic differential equation \((1.9)\), with \(\mu = 0.15\), \(\sigma = 0.1\) and \(X_0 = 95\).

1.1.6 Ornstein–Uhlenbeck Process

Many financial quantities, volatility amongst them, are modeled as mean-reverting processes, a term we shall explain in more detail in Chapters 2 and 3. The simplest example of a mean-reverting diffusion is the Ornstein–Uhlenbeck process, defined as a solution of

\[
dY_t = \alpha(m - Y_t)dt + \beta dW_t, \tag{1.20}
\]

where \(\alpha\) and \(\beta\) are positive constants. This is one of the few explicitly solvable stochastic differential equations, which we illustrate here as an application of Itô’s formula.

First, we re-arrange the terms to write

\[
dY_t + \alpha Y_t dt = \alpha m dt + \beta dW_t.
\]

Multiplying through by the “integrating factor” \(e^{\alpha t}\) gives

\[
d(e^{\alpha t}Y_t) = \alpha e^{\alpha t} dt + \beta e^{\alpha t} dW_t,
\]

where the left-hand exact integral is easily checked from Itô’s formula \((1.15)\). Integrating from zero to \(t\) and multiplying through by \(e^{-\alpha t}\) gives

\[
Y_t = m + (y - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} dW_s, \tag{1.21}
\]

where \(y\) is its (assumed known) starting value.

From this representation, it follows that \(Y\) is a Gaussian process and the distribution of \(Y_t\) is \(\mathcal{N}(m + (y - m)e^{-\alpha t}, \beta^2(1 - e^{-2\alpha t}))\). Its long-run distribution, obtained as \(t \to \infty\), is \(\mathcal{N}(m, \beta^2/2\alpha)\), which does not depend on \(y\). The concept of long-run (invariant) distribution will be discussed in detail in Chapter 3.
1.2 Derivative Contracts

Derivatives are contracts based on the underlying asset price \((X_t)\). They are also called *contingent claims*. We will be interested primarily in *options*, which can be European, American, path-independent or path-dependent. The definition of the options discussed in this first chapter is given in the following sections.

1.2.1 European Call and Put Options

A **European call option** is a contract that gives its holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined *strike price* \(K\) on the *maturity date* \(T\). If \(X_T\) is the price of the underlying asset at maturity time \(T\), then the value of this contract at maturity, its *payoff*, is

\[
h(X_T) = (X_T - K)^+ = \begin{cases} X_T - K & \text{if } X_T > K, \\ 0 & \text{if } X_T \leq K, \end{cases}
\]  

(1.22)

since in the first case the holder will exercise the option and make a profit \(X_T - K\) by buying the stock for \(K\) and selling it immediately at the market price \(X_T\). In the second case the option is not exercised, since the market price of the asset is less than the strike price.

Similarly, a **European put option** is a contract that gives its holder the right, but not the obligation, to sell a unit of the asset for a strike price \(K\) at the maturity date \(T\). Its payoff is

\[
h(X_T) = (K - X_T)^+ = \begin{cases} K - X_T & \text{if } X_T < K, \\ 0 & \text{if } X_T \geq K, \end{cases}
\]  

(1.23)

since in the first case buying the stock at the market price and exercising the put option yields a profit of \(K - X_T\), and in the second case the option is simply not exercised.

More generally, we will consider European derivatives defined by their maturity time \(T\) and their nonnegative payoff function \(h(x)\). This will be a contract which pays \(h(X_T)\) at maturity time \(T\) when the stock price is \(X_T\). The standard European-style derivatives are *path-independent* because the payoff \(h(X_T)\) is only a function of the value of the stock price at maturity time \(T\).

At time \(t < T\) this contract has a value, known as the *derivative price*, which will vary with \(t\) and the observed stock price \(X_t\). This option price at time \(t\) and for a stock price \(X_t = x\) is denoted by \(P(t, x)\) and the problem of *derivative pricing* is determining this pricing function. The fact that this option price will depend only on the observed value at time \(t\) and not on the past values of the stock price is closely related to the *Markov property* shared by the solutions of stochastic differential equations like (1.11), by which we shall model the stock price. More details on this will be given in Section 1.5.

Perhaps the simplest way to price such a derivative is as the expected value of its discounted payoff. More precisely, if the stock price is the process (1.2) and the observed stock price \(X_0 = x\), the option price at time \(t = 0\) would be

\[
P(0, x) = \mathbb{E} \left\{ e^{-rT} h(X_T) \right\} = \mathbb{E} \left\{ e^{-rT} h \left( x e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} \right) \right\},
\]  

(1.24)

where we have used the explicit formula (1.19) with \(X_0 = x\) and time \(T\). The expectation reduces to a Gaussian integral since \(W_T\) is \(\mathcal{N}(0, T)\)-distributed. In general (unless \(\mu = r\)) the option price given by formula (1.24) leads to an *arbitrage opportunity*, meaning that there will be a risk-free way to make a profit with strictly positive expectation by holding a particular portfolio. This is one of the key ideas in Section 1.3 that is used to determine the *fair* option price.
1.2.2 American Options

An American option is a contract in which the holder decides whether to exercise the option or not at any time of his choice before the option’s expiration date $T$. The time $\tau$ at which the option is exercised is called the exercise time. Because the market cannot be anticipated, the holder of the option has to decide to exercise or not at time $t \leq T$ with information up to time $t$ contained in the $\sigma$-algebra $\mathcal{F}_t$. In other words, $\tau$ is a random time such that the event $\{\tau \leq t\}$ (or its complement $\{\tau > t\}$) belongs to $\mathcal{F}_t$ for any $t \leq T$. Such a random time is called a stopping time with respect to the filtration $(\mathcal{F}_t)$. If the payoff function of the derivative is $h$, then its value at the exercise time $\tau$ is $h(X_\tau)$ where $X_\tau$ is the stock price at the stopping time $\tau$.

For an American call option the payoff is $h(X_\tau) = (X_\tau - K)^+$ for a given strike price $K$ and a stopping time $\tau \leq T$ chosen by the holder of the option. Note that even if $\tau$ is an exercise time the option will be exercised only if $X_\tau > K$ but, in any case, the contract is terminated at time $\tau$.

Similarly, the payoff of an American put option is $h(X_\tau) = (K - X_\tau)^+$ and the option is exercised only if $K > X_\tau$.

As in the case of European derivatives, an intuitive way to price an American derivative at time $t = 0$ is to maximize the expected value of the discounted payoff over all the stopping times $\tau \leq T$:

$$P(0, x) = \sup_{\tau \leq T} \mathbb{E} \left\{ e^{-r\tau} h(X_\tau) \right\}. \quad (1.26)$$

Again, this price leads in general to an opportunity for arbitrage and therefore cannot be the fair price of the derivative.

1.2.3 Other Exotic Options

The term “exotic option” refers here to any option contract which is not a standard European or American option described in the previous sections. Our aim is not to write a catalogue of existing options but rather to give some examples of exotic options that we will use in the rest of the book.

Barrier options are path-dependent options whose payoff depends on whether or not the underlying asset price hits a specified value during the option’s lifetime. For instance a down-and-out call option becomes worthless, or knocked out, if, at any time $t$ before the expiration date $T$, the stock price $X_t$ falls below a predetermined level $B$. The payoff at expiration $T$ is a function of the trajectory of the stock price

$$h = (X_T - \inf_{t \leq T} X_t)^+. \quad (1.27)$$

Here $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x$ is not in $A$. It is the indicator function of the set $A$. This option is obviously less valuable than a standard European call option given by (1.22) with the same strike $K$ and maturity $T$, and it will lead to a knock-out discount.

Lookback options are path-dependent options whose payoff functions depend on the minimum or maximum price of the underlying asset during the lifetimes of the options. In particular, a standard lookback call option has a payoff at maturity given by

$$h = (X_T - \inf_{t \leq T} X_t)^+ = X_T - \inf_{t \leq T} X_t, \quad (1.28)$$

where the lowest price plays the role of a floating strike price. Similarly a standard lookback put option has a payoff given by

$$h = \left( \sup_{t \leq T} X_t - X_T \right)^+ = \sup_{t \leq T} X_t - X_T, \quad (1.29)$$
where the highest price plays the role of the floating strike price. Note that these options are not genuine option contracts since they are almost always exercised, since \( h > 0 \) (IP a.s.) as can be seen from (1.28) and (1.29).

A **forward-start** or **cliquet** option is like a call option for instance, where the strike price is set at a later time. If \( t < T_1 < T \), then the payoff at maturity \( T \) is given by

\[
h = (X_T - X_{T_1})^+,
\]

where the stock price at time \( T_1 \) becomes the strike price.

**Compound options** are options on options. For instance, a *call-on-call* is the right to buy a call option at a later time for a predetermined price. It is an option written on an underlying which is itself an option with longer maturity. If \( t < T_1 < T \), then the payoff at maturity \( T_1 \) is given by

\[
h = (C_{T_1}(K, T) - K_1)^+,
\]

where \( C_{T_1}(K, T) \) is the price at time \( T_1 \) of a call option which pays \( (X_T - K)^+ \) at maturity time \( T \).

Our last example is **Asian options** whose payoff depends on the average stock price during a specified period of time before maturity. They can be European or American with typical payoffs like

\[
h = \left( X_T - \frac{1}{T} \int_0^T X_s ds \right)^+,
\]

for an *arithmetic-average strike call option* (European style), where the strike price is the average stock price.

### 1.3 Replicating Strategies

The Black–Scholes analysis of a European style derivative yields an explicit trading strategy in the underlying risky asset and riskless bond whose terminal payoff is equal to the payoff \( h(X_T) \) of the derivative at maturity, no matter what path the stock price takes. Thus, selling the derivative and holding a dynamically adjusted portfolio according to this strategy “covers” an investor against all risk of eventual loss, because a loss incurred at the final time from one part of this portfolio will be exactly compensated by a gain in the other part. This replicating strategy, as it is known, therefore provides an insurance policy against the risk of being short the derivative. It is called a *dynamic hedging strategy*, since it involves continuous trading, where to hedge means to eliminate risk. The essential step in the Black–Scholes methodology is the construction of this replicating strategy and arguing, based on no-arbitrage, that the value of the replicating portfolio at time \( t \) is the fair price of the derivative. We develop this argument in the following sections.

#### 1.3.1 Replicating Self-Financing Portfolios

We consider a European style derivative with payoff \( h(X_T) \), a function of the underlying asset price at maturity time \( T \). Assume that the stock price \( (X_t) \) follows the geometric Brownian motion model (1.19), solution of the stochastic differential equation (1.2). A **trading strategy** is a pair \((a_t, b_t)\) of adapted processes specifying the number of units held at time \( t \) of the underlying asset and the riskless bond, respectively. We suppose that \( \mathbb{E} \left\{ \int_0^T a_t^2 dt \right\} \) and \( \int_0^T |b_t| dt \) are finite so that the stochastic integral involving \((a_t)\) and the usual integral involving \((b_t)\) are well-defined.
Assuming, as in (1.1), that the price of the bond at time \( t \) is \( \beta_t = e^{rt} \), the value at time \( t \) of this portfolio is \( a_t X_t + b_t e^{rt} \). It will replicate the derivative at maturity if its value at time \( T \) is almost surely equal to the payoff:

\[
a_T X_T + b_T e^{rT} = h(X_T). \tag{1.33}
\]

In addition, this portfolio is to be self-financing, meaning that the variations of its value are due only to the variations of the market - that is, the variations of the stock and bond prices. No further funds are required after the initial investment so that if, for example, more of the asset is bought (\( a_t \) is increased), then money would have to be obtained by selling bonds (\( b_t \) decreased) to pay for it. This is expressed in differential form as

\[
d \left( a_t X_t + b_t e^{rt} \right) = a_t dX_t + rb_t e^{rt} \, dt. \tag{1.34}
\]

In integral form, the self-financing property is

\[
a_t X_t + b_t e^{rt} = a_0 X_0 + b_0 + \int_0^t a_s dX_s + \int_0^t r b_s e^{rs} \, ds, \quad 0 \leq t \leq T.
\]

An intuitive way to understand this relation is to think in terms of discrete trading times \( \{t_n, n = 0, 1, \ldots \} \). The portfolio consists of \( a_{t_n} \) and \( b_{t_n} \) of the stock and bond (respectively) at time \( t_n \). When the prices change to \( X_{t_{n+1}} \) and \( e^{r t_{n+1}} \), we observe the change and then we adjust our holdings to \( a_{t_{n+1}} \) and \( b_{t_{n+1}} \). As no further cash input or output is allowed, the value of the portfolio after adjustment must equal the value before, so

\[
a_{t_n} X_{t_{n+1}} + b_{t_n} e^{r t_{n+1}} = a_{t_{n+1}} X_{t_{n+1}} + b_{t_{n+1}} e^{r t_{n+1}}.
\]

This says that

\[
a_{t_{n+1}} X_{t_{n+1}} + b_{t_{n+1}} e^{r t_{n+1}} - \left( a_{t_n} X_{t_n} + b_{t_n} e^{r t_n} \right) = a_{t_n} \left( X_{t_{n+1}} - X_{t_n} \right) + b_{t_n} \left( e^{r t_{n+1}} - e^{r t_n} \right),
\]

which in continuous time becomes (1.34).

### 1.3.2 The Black–Scholes Partial Differential Equation

As in Section 1.2.1, the pricing function for a European-style contract with payoff \( h(X_T) \) is denoted by \( P(t, x) \). At this stage we do not know that we can find such a function relating the option price only to the present risky asset price and not to its history. Nevertheless, we shall assume that such a pricing function \( P(t, x) \) exists and is regular enough to apply Itô’s formula (1.16). Our goal is to construct a self-financing portfolio \( (a_t, b_t) \) that will replicate the derivative at maturity (1.33).

The no-arbitrage condition requires that

\[
a_t X_t + b_t e^{rt} = P(t, X_t), \quad \text{for any } 0 \leq t \leq T. \tag{1.35}
\]

For if at some time \( t < T \) the left-hand side of (1.35) is (say) less than the right-hand side, an arbitrage opportunity exists by selling the over-priced derivative security immediately and investing in the under-priced asset-bond trading strategy, yielding an instant profit with no exposure to future loss since the terminal payoff of the trading strategy is equal to the payoff of the derivative.

Differentiating (1.35) and using the self-financing property (1.34) on the left-hand side, Itô’s formula (1.16) on the right-hand side and equation (1.2), we obtain

\[
\left( a_t \mu X_t + b_t e^{rt} \right) dt + a_t \sigma X_t dW_t = \left( \frac{\partial P}{\partial t} + \mu X_t \frac{\partial P}{\partial x} + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt + \sigma X_t \frac{\partial P}{\partial x} dW_t, \tag{1.36}
\]
where all the partial derivatives of $P$ are evaluated at $(t, X_t)$. Equating the coefficients of the $dW_t$ terms gives

$$a_t = \frac{\partial P}{\partial x}(t, X_t).$$

(1.37)

From (1.35) we get

$$b_t = (P(t, X_t) - a_t X_t) e^{-rt}.$$

(1.38)

Equating the $dt$ terms in (1.36) gives

$$r \left( P - X_t \frac{\partial P}{\partial x} \right) = \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2},$$

(1.39)

which is satisfied for any stock price $X_t$ if $P(t, x)$ is the solution of the Black–Scholes partial differential equation

$$\mathcal{L}_{BS}(\sigma) P = 0,$$

(1.40)

where the Black–Scholes operator is defined by

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right).$$

(1.41)

Equation (1.40) holds in the domain $t \leq T$ and $x > 0$, since in our model the stock price remains positive. It is to be solved backward in time with the final condition $P(T, x) = h(x)$, because at expiration the price of the derivative is simply its payoff.

The partial differential equation (1.40) with its final condition has a unique solution $P(t, x)$, which is the value of a self-financing replicating portfolio. Knowing $P$, the portfolio $(a_t, b_t)$ is uniquely determined by (1.37) and (1.38).

Surprisingly, the rate of return $\mu$ does not enter at all in the valuation of this portfolio, owing to the cancellation of the $\mu$ terms in the equation (1.36) after the determination (1.37) of $a_t$. This is a key fact in the setting of the fair price of the derivative as an expected payoff given in Section 1.4. It is a remarkable feature of the Black–Scholes theory that if two investors have different speculative views about the growth rate of the risky asset - meaning that they have different values of $\mu$ but agree that the (commonly-estimated and stable) historical volatility $\sigma$ will prevail - then they still agree on the no-arbitrage price of the derivative, since $P$ does not depend on $\mu$.

1.3.3 Pricing to Hedge

There is another way to derive the Black–Scholes partial differential equation that emphasizes risk elimination or hedging. It is a reinterpretation of the calculations in the previous section as follows.

Let $P_t = P(t, X_t)$ be the price of the option. If we sell $N_t$ options and hold $A_t$ units of the risky asset $X_t$, then the change in the value of this portfolio is $A_t dX_t - N_t dP_t$ because it is assumed to be self-financing. We now determine $(A, N)$ so that this portfolio is riskless, which means that we set to zero the coefficient of $dW_t$. The change in the value of the portfolio should then equal that of a riskless asset, so

$$A_t dX_t - N_t dP_t = r(A_t X_t - N_t P_t) dt.$$
Using (1.9) and Itô’s formula we have

$$A_t(\mu X_t dt + \sigma X_t dW_t) = N_t \left\{ \left( \frac{\partial P}{\partial t} + \mu X_t \frac{\partial P}{\partial x} + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt - \sigma X_t \frac{\partial P}{\partial x} dW_t \right\}$$

$$= r(A_t X_t - N_t P_t) dt.$$  

Eliminating the $dW_t$ terms gives

$$A_t = N_t \frac{\partial P}{\partial x}(t, X_t),$$

and then the terms involving $\mu$ cancel also. We are thus left with the Black–Scholes partial differential equation (1.40) for $P(t, x)$.

In this derivation of the Black–Scholes pricing equation, the role of hedging is clear. Selling the option and holding a dynamically adjusted amount of the risky asset so as to eliminate risk determines the price of the option $P_t$ and the hedge ratio $A_t/N_t$. This is known as Delta Hedging and the ratio $a_t = A_t/N_t$ given by (1.37) is called the Delta.

### 1.3.4 The Black–Scholes Formula

For European call options described in Section 1.2.1, the Black–Scholes partial differential equation (1.40) is solved with the final condition $h(x) = (x - K)^+$. Prices of European calls at time $t$ and for an observed risky asset price $X_t = x$ will be denoted by $C_{BS}(t, x)$. In this particular case, there is a closed-form solution known as the Black–Scholes formula:

$$C_{BS}(t, x) = x N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where

$$d_1 = \log(x/K) + \left( r + \frac{1}{2} \sigma^2 \right) (T - t) \sqrt{T - t},$$

$$d_2 = d_1 - \sigma \sqrt{T - t},$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy,$$

is the standard Gaussian cumulative distribution function. The probabilistic derivation of the Black–Scholes formula (1.42) is given in Section 1.4.4.

This convenient formula for the price of a call option - in terms of the current stock price $x$, the time-to-maturity $T - t$, the strike price $K$, the volatility $\sigma$, and the short rate $r$ - explains the popularity of the model in the financial services industry since the mid-1970s. We will also denote $C_{BS}$ by $C_{BS}(t, x; K, T; \sigma)$ to emphasize the dependence on $K, T$ and $\sigma$. Only the volatility $\sigma$, the standard deviation of the returns scaled by the square root of the time increment, needs to be estimated from data, assuming that the constant short rate $r$ is known.

The fact that $C_{BS}(t, x)$ given by (1.42) satisfies equation (1.40) with the final condition $h(x) = (x - K)^+$ can easily be checked directly. A probabilistic representation of this solution is presented in the following section.

Figure 1.2 shows the pricing function $C_{BS}(0, x; 100, 0.5; 0.1)$ plotted against the present ($t = 0$) stock price $x$. Notice how it is a smoothed version of the “ramp” terminal payoff function.
The Delta hedging ratio \( a_t \) for a call is given by
\[
\frac{\partial C_{BS}}{\partial x} = N(d_1) .
\]

There is a similar formula for European put options. Let \( P_{BS}(t, x) \) be the price of a European put option (Section 1.2.1). We then have the put-call parity relation
\[
C_{BS}(t, X_t) - P_{BS}(t, X_t) = X_t - Ke^{-r(T-t)} ,
\]
(1.46)
between put and call options with the same maturity and strike price. This is a model-free relationship that follows from simple no-arbitrage arguments. If, for instance, the left side is smaller than the right side then buying a call and selling a put and one unit of the stock, and investing the difference in the bond, creates a profit no matter what the stock price does.

Under the lognormal model, this relationship can be checked directly since the difference \( C_{BS} - P_{BS} \) satisfies the partial differential equation (1.40) with the final condition \( h(x) = x - K \). This problem has the unique simple solution \( x - Ke^{-r(T-t)} \). Using the Black–Scholes formula (1.42) for \( C_{BS} \) and the put-call parity relation (1.46), we deduce the following explicit formula for the price of a European put option:
\[
P_{BS}(t, x) = Ke^{-r(T-t)}N(-d_2) - xN(-d_1) ,
\]
(1.47)
where \( d_1, d_2 \) and \( N \) are as in (1.43), (1.44) and (1.45) respectively.

Figure 1.3 shows the pricing function \( P_{BS}(0, x; 100, 0.5; 0.1) \) plotted against the present \( (t = 0) \) stock price \( x \). Here we see that the pricing function crosses over its terminal payoff for some (small enough) \( x \), which does not happen with the call option function in Figure 1.2. This observation will be important when we will look at American options in Section 1.6.
Other types of options do not lead in general to such explicit formulas. Determining their prices requires solving numerically the partial differential equation (1.40) with appropriate boundary conditions. Nevertheless, probabilistic representations can be obtained as explained in the following section. In particular, American options lead to free-boundary value problems associated with equation (1.40).

1.3.5 The Greeks

The sensitivity of the price of an option to the variations of its parameters is measured by using partial derivatives. These quantities, known as Greeks of the option, play an important role in the derivative markets. For instance, we have already introduced in the previous section the Delta of a call option:

\[
\Delta_{BS} = \frac{\partial C_{BS}}{\partial x} = N(d_1).
\]  

(1.48)

The second derivative with respect to the stock price is also known as the Gamma which, in the case of a call option, is given by:

\[
\Gamma_{BS} = \frac{\partial^2 C_{BS}}{\partial x^2} = \frac{\partial \Delta_{BS}}{\partial x} = \frac{e^{-d_1^2/2}}{x\sigma \sqrt{2\pi(T-t)}}.
\]

(1.49)

The sensitivity to the volatility level, represented by the first derivative with respect to \( \sigma \), is called the Vega:

\[
\nu_{BS} = \frac{\partial C_{BS}}{\partial \sigma} = \frac{xe^{-d_1^2/2}}{\sqrt{2\pi(T-t)}}.
\]

(1.50)
The sensitivities with respect to time to maturity $T - t$ and short rate $r$ are respectively named the Theta and the Rho.

In the general case of an European derivative whose price satisfies the Black–Scholes partial differential equation (1.40) with a terminal condition $P(T, x) = h(x)$, there are simple and important relations between some of the Greeks. These are obtained by differentiating (1.40). For instance, differentiating with respect to $\sigma$ leads to the following equation for the Vega:

$$L_{BS}(\sigma)V + \sigma x^2 \partial^2 P \partial x^2 = 0,$$

with a zero terminal condition. One can easily check that the Black–Scholes operator $L_{BS}(\sigma)$ commutes with $x^2 \partial^2 / \partial x^2$, and therefore that $(T - t)\sigma x^2 \partial^2 P / \partial x^2$ satisfies equation (1.51). If the second derivative with respect to $x$ remains bounded as $t \to T$, this solution satisfies the zero terminal condition, and therefore we obtain the following relation between the Vega and the Gamma

$$\frac{\partial P}{\partial \sigma} = (T - t)\sigma x^2 \partial^2 P \partial x^2.$$  

(1.52)

In the case of a call option this relation can be directly obtained from (1.49) and (1.50). Using the same argument, by differentiating the Black–Scholes equation with respect to $r$, one can obtain the relation

$$\frac{\partial P}{\partial r} = (T - t)\left( x \frac{\partial P}{\partial x} - P \right).$$

(1.53)

It is important to note that these relations may not be satisfied by more complex derivatives involving additional boundary conditions, such as barrier options for instance.

### 1.4 Risk-Neutral Pricing

We mentioned in Section 1.2.1 that, unless $\mu = r$, the expected value under the real-world probability $\mathcal{P}$ of the discounted payoff of a derivative (1.24) would lead to an opportunity for arbitrage. This is closely related to the fact that the discounted price $\tilde{X}_t = e^{-rt}X_t$ is not a martingale since, from (1.18),

$$d\tilde{X}_t = (\mu - r)\tilde{X}_tdt + \sigma\tilde{X}_tdW_t,$$

which contains a nonzero drift term if $\mu \neq r$.

The main result we want to build in this section is that there is a unique probability measure $\mathcal{P}^*$ equivalent to $\mathcal{P}$ such that, under this probability, (i) the discounted price $\tilde{X}_t$ is a martingale and (ii) the expected value under $\mathcal{P}^*$ of the discounted payoff of a derivative gives its no-arbitrage price. Such a probability measure describing a risk-neutral world is called an equivalent martingale measure.

#### 1.4.1 Equivalent Martingale Measure

In order to find a probability measure under which the discounted price $\tilde{X}_t$ is a martingale, we rewrite (1.54) in such a way that the drift term is “absorbed” in the martingale term:

$$d\tilde{X}_t = \sigma\tilde{X}_t dw + \left( \frac{\mu - r}{\sigma} \right) \tilde{X}_t dt.$$
We set
\[
\theta = \frac{\mu - r}{\sigma},
\]
called the *market price of asset risk*, and we define
\[
W^*_t = W_t + \int_0^t \theta \, ds = W_t + \theta t,
\]
so that
\[
\bar{\mathcal{N}}_t = \sigma \bar{X}_t dW^*_t.
\]
Using the characterization (1.3), it is easy to check that the positive random variable \(\xi^\theta_T\) defined by
\[
\xi^\theta_T = \exp \left( -\theta W_T - \frac{1}{2} \theta^2 T \right),
\]
has an expected value (with respect to \(\mathbb{P}\)) equal to 1 (the Cameron–Martin formula). More generally, it has a conditional expectation with respect to \(\mathcal{F}_t\) given by
\[
\mathbb{E}\{\xi^\theta_T \mid \mathcal{F}_t\} = \exp \left( -\theta W_t - \frac{1}{2} \theta^2 t \right) = \xi^\theta_t, \quad \text{for } 0 \leq t \leq T,
\]
which defines a martingale denoted by \(\xi^\theta_t\) \(0 \leq t \leq T\).

We now introduce the probability measure \(\mathbb{P}^*\). It is the equivalent measure to \(\mathbb{P}\), meaning that they have the same null sets (they agree on which events have zero probability), which has the density \(\xi^\theta_T\) with respect to \(\mathbb{P}\):
\[
d\mathbb{P}^* = \xi^\theta_T d\mathbb{P}.
\]
Denoting by \(\mathbb{E}^*\{\cdot\}\) the expectation with respect to \(\mathbb{P}^*\), for any integrable random variable \(Z\) we have
\[
\mathbb{E}^*\{Z\} = \mathbb{E}\{\xi^\theta_T Z\},
\]
and one can check that, for any adapted and integrable process \((Z_t)\),
\[
\mathbb{E}^*\{Z_t \mid \mathcal{F}_s\} = \frac{1}{\xi^\theta_s} \mathbb{E}\{\xi^\theta_s Z_t \mid \mathcal{F}_s\},
\]
for any \(0 \leq s \leq t \leq T\). The process \((\xi^\theta_t)_{0 \leq t \leq T}\) is called the Radon–Nikodym process.

The main result of this section asserts that the process \((W^*_t)\) given by (1.56) is a standard Brownian motion under the probability \(\mathbb{P}^*\). This result in its full generality (when \(\theta\) is an adapted stochastic process) is known as Girsanov’s Theorem. In our simple case (\(\theta\) constant), it is easily derived by using the characterization (1.3) and formula (1.60) as follows:
\[
\mathbb{E}^*\{e^{iu(W^*_t-W^*_s)} \mid \mathcal{F}_s\} = \frac{1}{\xi^\theta_s} \mathbb{E}\{\xi^\theta_s e^{iu(W^*_t-W^*_s)} \mid \mathcal{F}_s\} \\
= e^{\theta W_s + \frac{1}{2} \theta^2 s} \mathbb{E}\{e^{-\theta W_t - \frac{1}{2} \theta^2 t} e^{iu(W_t-W_s+\theta(t-s))} \mid \mathcal{F}_s\} \\
= e^{(-\frac{1}{2} \theta^2 + iu \theta)(t-s)} \mathbb{E}\{e^{i(u+i \theta)(W_t-W_s)} \mid \mathcal{F}_s\} \\
= e^{(-\frac{1}{2} \theta^2 + iu \theta)(t-s)} e^{-\frac{1}{2}(u+i \theta)^2(t-s)} \\
= e^{-\frac{u^2(t-s)}{2}}.
\]
1.4.2 Self-Financing Portfolios

As in Section 1.3.1, a portfolio comprises \( a_t \) units of stock and \( b_t \) in bonds; we denote by \( V_t \) its value at time \( t \):

\[
V_t = a_t X_t + b_t e^{rt}.
\]

The self-financing property (1.34), namely \( dV_t = a_t dX_t + b_t e^{rt} dt \), implies that the discounted value of the portfolio, \( \tilde{V}_t = e^{-rt}V_t \), is a martingale under the risk-neutral probability \( \mathbb{P}^\star \). This essential property of self-financing portfolios is obtained as follows:

\[
d\tilde{V}_t = -re^{-rt} \tilde{V}_t dt + e^{-rt} dV_t
\]

\[
= -re^{-rt}(a_t X_t + b_t e^{rt}) dt + e^{-rt}(a_t dX_t + b_t e^{rt} dt)
\]

\[
= -re^{-rt}a_t X_t dt + e^{-rt}a_t dX_t
\]

\[
= a_t d(e^{-rt}X_t)
\]

\[
= a_t d\tilde{X}_t
\]

\[
= \sigma a_t \tilde{X}_t dW^\star_t \quad \text{(by (1.57))},
\]

which shows that \( \tilde{V}_t \) is a martingale under \( \mathbb{P}^\star \) as a stochastic integral with respect to the Brownian motion \( W^\star_t \). Indeed, the same computation shows that if a portfolio satisfies \( d\tilde{V}_t = a_t d\tilde{X}_t \), then it is self-financing.

A simple calculation demonstrates the connection between martingales and no-arbitrage. Suppose that \((a_t, b_t)_{0 \leq t \leq T}\) is a self-financing arbitrage strategy; that is,

\[
V_T \geq e^{rT} V_0 \quad (\mathbb{P}-\text{a.s.}),
\]

with

\[
\mathbb{P}\{V_T > e^{rT} V_0\} > 0,
\]

so that the strategy never makes less than money in the bank and there is some chance of making more. By the martingale property of self-financing strategies

\[
\mathbb{E}^\star\{V_T\} = e^{rT} V_0,
\]

so that (1.62) and (1.63) cannot both hold. This is because \( \mathbb{P} \) and \( \mathbb{P}^\star \) are equivalent and therefore, (1.62) and (1.63) also hold with \( \mathbb{P} \) replaced by \( \mathbb{P}^\star \). The inequality (1.62) (which holds \( \mathbb{P}^\star\)-a.s.) and (1.64) imply \( V_T = e^{rT} V_0 \) (\( \mathbb{P}^\star\)-a.s. and \( \mathbb{P}\)-a.s.), contradicting (1.63).

1.4.3 Risk-Neutral Valuation

Assume that \((a_t, b_t)\) is a self-financing portfolio satisfying the same integrability conditions of Section 1.3.1 and replicating the European style derivative with nonnegative payoff \( H \):

\[
a_T X_T + b_T e^{rT} = H,
\]

where we assume that \( H \) is a square integrable \( \mathcal{F}_T \)-adapted random variable. This includes European calls and puts or more general standard European derivatives for which \( H = h(X_T) \), as well as other European style exotic derivatives presented in Section 1.2.3.
On one hand, a no-arbitrage argument shows that the price at time $t$ of this derivative should be the value $V_t$ of this portfolio. On the other hand, as shown in Section 1.4.2, the discounted values ($\tilde{V}_t$) of this portfolio form a martingale under the risk-neutral probability $\mathbb{P}^*$ and consequently

$$\tilde{V}_t = \mathbb{E}^* \left\{ \tilde{V}_T \mid \mathcal{F}_t \right\},$$

which gives

$$V_t = \mathbb{E}^* \left\{ e^{-r(T-t)}h \left( X_T \right) \mid \mathcal{F}_t \right\},$$

after reintroducing the discounting factor and using the replicating property (1.65).

Alternatively, given the risk-neutral valuation formula (1.66), we can find a self-financing replicating portfolio for the payoff $H$. The existence of such a portfolio is guaranteed by an application of the martingale representation theorem: for $0 \leq t \leq T$

$$M_t = \mathbb{E}^* \left\{ e^{-rT}h \mid \mathcal{F}_t \right\},$$

defines a square integrable martingale under $\mathbb{P}^*$ with respect to the filtration ($\mathcal{F}_t$), which is also the natural filtration of the Brownian motion $W^*$. This representation theorem says that any such martingale is a stochastic integral with respect to $W^*$, so that

$$\mathbb{E}^* \left\{ e^{-rT}h \mid \mathcal{F}_t \right\} = M_0 + \int_0^t \eta_s dW^*_s,$$

where $(\eta_t)$ is some adapted process with $\mathbb{E}^* \left\{ \int_0^T \eta_t^2 dt \right\}$ finite. By defining $a_t = \eta_t / (\sigma \tilde{X}_t)$ and $b_t = M_t - a_t \tilde{X}_t$, we construct a portfolio $(a_t, b_t)$, which is shown to be self-financing by checking that its discounted value is the martingale $M_t$ and using the characterization (1.61) obtained in Section 1.4.2. Its value at time $T$ is $e^{rT}M_T = H$ and therefore it is a replicating portfolio.

### 1.4.4 Using the Markov Property

If $H$ is a function of the path of the stock price after time $t$ - as, for instance, for a standard European derivative with payoff $H = h(X_T)$ - then the Markov property of $(X_t)$ says that conditioning with respect to the past $\mathcal{F}_t$ is the same as conditioning with respect to $X_t$, the value at the current time; this gives

$$V_t = \mathbb{E}^* \left\{ e^{-r(T-t)}h(X_T) \mid X_t \right\}.$$

We will come back to this property in the next section.

Denoting by $P(t, x)$, as in Section 1.3.2, the price of this derivative at time $t$ for an observed stock price $X_t = x$, we obtain the pricing formula

$$P(t, x) = \mathbb{E}^* \left\{ e^{-r(T-t)}h(X_T) \mid X_t = x \right\}.$$

If we compare this formula (at time $t = 0$) with (1.24), our first intuitive idea for pricing a standard European derivative in Section 1.2.1, we see that the essential step is to replace the “real-world” $\mathbb{P}$ by the “risk-neutral world” $\mathbb{P}^*$ in order to obtain the fair no-arbitrage price.
Knowing that \( X_t = x \), one can generalize the formula (1.19) obtained in Section 1.1.5 and obtain an explicit formula for \( X_T \) by solving the stochastic differential equation (1.2) from \( t \) to \( T \) starting from \( x \):

\[
X_T = x \exp \left( (\mu - \frac{\sigma^2}{2})(T-t) + \sigma (W_T - W_t) \right). \tag{1.68}
\]

Using (1.56), this formula can be rewritten in terms of \((W_t^*)\) as

\[
X_T = x \exp \left( (r - \frac{\sigma^2}{2})(T-t) + \sigma (W_T^* - W_t^*) \right). \tag{1.69}
\]

As \((W_t^*)\) is a standard Brownian motion under the risk-neutral probability \( \mathbb{P}^* \), the increment \( W_T^* - W_t^* \) is \( \mathcal{N}(0, T-t) \)-distributed, and (1.67) gives the Gaussian integral

\[
P(t, x) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} e^{-r(T-t)h} \left( xe^{(r-\frac{\sigma^2}{2})(T-t)+\sigma z} \right) e^{-\frac{z^2}{2(T-t)}} dz. \tag{1.70}
\]

In the case of a European call option, \( h(x) = (x-K)^+ \), this integral reduces to the Black–Scholes formula (1.42) obtained in Section 1.3.4, as the following computation shows:

\[
P(t, x) = \frac{x}{\sqrt{2\pi T}} \int_{z^*}^{+\infty} e^{-\frac{(z^*-\sigma^2)^2}{2\tau}} dz - \frac{K e^{-r\tau}}{\sqrt{2\pi T}} \int_{z^*}^{+\infty} e^{-\frac{z^2}{2\tau}} dz,
\]

where \( \tau = T-t \) and \( z^* \) is defined by \( x \exp \left( (r - \frac{1}{2}\sigma^2)\tau + \sigma z^* \right) = K \). We then set

\[
\frac{z^* - \sigma\tau}{\sqrt{\tau}} = -d_1, \quad \frac{z^*}{\sqrt{\tau}} = -d_2,
\]

which coincide with the definitions (1.43) and (1.44) of \( d_1 \) and \( d_2 \). The Black–Scholes formula (1.42) follows by introducing the Gaussian cumulative distribution function \( N \) given by (1.45).

Another important example is given by the binary or digital options which, for instance, pays at time \( T \) a fixed amount (say one), if \( X_T \geq K \), and nothing otherwise. The corresponding discontinuous payoff function is simply \( h(x) = 1_{\{x \geq K\}} \). Its value at time \( t \) is given by (1.67), which, in this case, becomes

\[
P_{digital}(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi \tau}} \int_{z^*}^{+\infty} e^{-\frac{z^2}{2\tau}} dz = e^{-r\tau} N(d_2). \tag{1.70}
\]

The two approaches developed in Sections 1.3 and 1.4 should give the same fair price to the same derivative. This is indeed the case, and is the content of the following section, where we explain that a formula like (1.67) is just a probabilistic representation of the solution of a partial differential equation like (1.40).

### 1.5 Risk-Neutral Expectations and Partial Differential Equations

In Section 1.4.4, we used the Markov property of the stock price \((X_t)\) and, in order to compute \( X_T \) knowing that \( X_t = x \) at time \( t \leq T \), we solved the stochastic differential equation (1.2) between \( t \) and \( T \). This was a particular case of the general situation where \((X_t)\) is the unique solution of
the stochastic differential equation (1.11). We denote by \((X^{t,x}_s)_{s \geq t}\) the solution of that equation starting from \(x\) at time \(t\):

\[
X^{t,x}_s = x + \int_t^s \mu(u, X^{t,x}_u) \, du + \int_t^s \sigma(u, X^{t,x}_u) \, dW_u,
\]

and we assume enough regularity in the coefficients \(\mu\) and \(\sigma\) for \((X^{t,x}_s)\) to be jointly continuous in the three variables \((t, x, s)\). The flow property for deterministic differential equations can be extended to stochastic differential equations like (1.71); it says that, in order to compute the solution at time \(s > t\) starting at time 0 from point \(x\), one can use

\[
x \rightarrow X^{0,x}_t \rightarrow X^{t,x}_s = X^{0,x}_s \quad (\mathbb{P}\text{-a.s.}).
\]

In other words, one can solve the equation from 0 to \(t\), starting from \(x\), to obtain \(X^{0,x}_t\). Then we solve the equation from \(t\) to \(s\), starting from \(X^{0,x}_t\). This is the same as solving the equation from 0 to \(s\), starting from \(x\).

The Markov property is a consequence and can be stated as follows:

\[
\mathbb{E}\{h(X_s) \mid \mathcal{F}_t\} = \mathbb{E}\{h(X^{t,x}_s)\} \mid x = X_t,
\]

which is what we have used with \(s = T\) to derive (1.67). Observe that the discounting factor could be pulled out of the conditional expectation since the interest rate is constant (not random). In the time homogeneous case (\(\mu\) and \(\sigma\) independent of time) we further have

\[
\mathbb{E}\{h(X^{t,x}_s)\} = \mathbb{E}\{h(X^{0,x}_{s-t})\},
\]

which could have been used with \(s = T\) to derive (1.69) since \(W^*_T\) is \(\mathcal{N}(0, T - t)\)-distributed.

### 1.5.1 Infinitesimal Generators and Associated Martingales

For simplicity, we first consider a time homogeneous diffusion process \((X_t)\), solution of the stochastic differential equation

\[
dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t.
\]

Let \(g\) be a twice continuously differentiable function of the variable \(x\) with bounded derivatives, and define the differential operator \(\mathcal{L}\) acting on \(g\) according to

\[
\mathcal{L}g(x) = \frac{1}{2} \sigma^2(x) g''(x) + \mu(x) g'(x).
\]

In terms of \(\mathcal{L}\), Itô’s formula (1.16) gives

\[
dg(X_t) = \mathcal{L}g(X_t) \, dt + g'(X_t) \sigma(X_t) \, dW_t,
\]

which shows that

\[
M_t = g(X_t) - \int_0^t \mathcal{L}g(X_s) \, ds,
\]

defines a martingale. Consequently, if \(X_0 = x\), we obtain

\[
\mathbb{E}\{g(X_t)\} = g(x) + \mathbb{E}\left\{\int_0^t \mathcal{L}g(X_s) \, ds\right\}.
\]
Under the assumptions made on the coefficients $\mu$ and $\sigma$ and on the function $g$, the Lebesgue dominated convergence theorem is applicable and gives

\[
\frac{d}{dt} \mathbb{E}\{g(X_t)\}_{t=0} = \lim_{t \to 0} \frac{\mathbb{E}\{g(X_t)\} - g(x)}{t} = \lim_{t \to 0} \mathbb{E} \left\{ \frac{1}{t} \int_0^t \mathcal{L}g(X_s) ds \right\} = \mathcal{L}g(x).
\]

The differential operator $\mathcal{L}$ given by (1.75) is called the infinitesimal generator of the Markov process $(X_t)$.

Considering now a nonhomogeneous diffusion $(\sigma(t, x), \mu(t, x))$ and functions $g(t, x)$ which depend also on time, (1.76) can be generalized by using the full Itô formula (1.16) to yield the martingale

\[
M_t = g(t, X_t) - \int_0^t \left( \frac{\partial g}{\partial t} + \mathcal{L}g \right) (s, X_s) ds,
\]

where the infinitesimal generator $\mathcal{L}_t$ is defined by

\[
\mathcal{L}_t = \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2} + \mu(t, x) \frac{\partial}{\partial x},
\]

and $g$ is any smooth and bounded function. Finally, it is possible to incorporate a discounting factor by using the integration-by-parts formula to compute the differential of $e^{-rt}g(t, X_t)$ and obtain the martingales

\[
M_t = e^{-rt}g(t, X_t) - \int_0^t e^{-rs} \left( \frac{\partial g}{\partial t} + \mathcal{L}g - rg \right) (s, X_s) ds,
\]

which introduces the potential term $-rg$. This can also be generalized to the case of a potential depending on $t$ and $x$, $e^{-rt}$ being replaced by the discounting factor $e^{-\int_0^t r(s, X_s) ds}$.

### 1.5.2 Conditional Expectations and Parabolic Partial Differential Equations

Suppose that $u(t, x)$ is a solution of the partial differential equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 u}{\partial x^2} + \mu(t, x) \frac{\partial u}{\partial x} - ru = 0,
\]

with the final condition $u(T, x) = h(x)$ and assume that it is regular enough to apply Itô’s formula (1.16). Using (1.79) we deduce that $M_t = e^{-rt}u(t, X_t)$ is a martingale when $\mathcal{L}_t$, given by (1.78), is the infinitesimal generator of the process $(X_t)$ - in other words, when $\mu$ and $\sigma$ are the drift and diffusion coefficients of $(X_t)$.

The martingale property for times $t$ and $T$ reads $\mathbb{E}\{M_T \mid F_t\} = M_t$ which can be rewritten as

\[
u(t, X_t) = \mathbb{E} \left\{ e^{-r(T-t)} h(X_T) \mid F_t \right\},
\]

since $u(T, X_T) = h(X_T)$ according to the final condition. Using the Markov property (1.73), we deduce the following probabilistic representation of the solution $u$:

\[
u(t, x) = \mathbb{E} \left\{ e^{-r(T-t)} h(X_T^{t,x}) \right\},
\]

which may also be written as

\[
u(t, x) = \mathbb{E} \left\{ e^{-r(T-t)} h(X_T) \mid X_t = x \right\} \quad \text{or} \quad \nu(t, x) = \mathbb{E}_{t,x} \left\{ e^{-r(T-t)} h(X_T) \right\}.
\]

If $r$ depends on $t$ and $x$, the discounting factor becomes $e^{-\int_t^T r(s, X_s) ds}$. The representation (1.81) is then called the Feynman–Kac formula.
1.5.3 Kolmogorov Equations

When $r = 0$, equation (1.80) reads

$$\frac{\partial u}{\partial t} + \mathcal{L}_t u = 0, \quad u(T, x) = h(x),$$

(1.82)

with $\mathcal{L}_t$ given in (1.78). This equation is known as the backward Kolmogorov equation for the conditional expectation. Let $p(t, x; T, \xi)$ denote the transition probability density of $X_T$ starting at time $t$ at point $x$, and assume that it is sufficiently regular for the following calculations. Then

$$u(t, x) = \mathbb{E} \{ h(X_T) \mid X_t = x \} = \int h(\xi') p(t, x; T, \xi') d\xi',$$

(1.83)

and by choosing $h(x) = \delta(x - \xi)$, the point mass or Dirac delta function at $\xi$, gives $u(t, x) = p(t, x; T, \xi)$, and therefore

$$\frac{\partial p}{\partial t} + \mathcal{L}_t p = 0, \quad p(T, x; T, \xi) = \delta(x - \xi),$$

where the operator $\mathcal{L}_t$ acts on the variable $x$. This equation is known as the backward Kolmogorov equation for the transition probability density.

It is also useful to have an equation for $p$ with respect to its forward variables $(T, \xi)$. To this end, we define the adjoint operator $\mathcal{L}_T^*$ of $\mathcal{L}_T$ by

$$\int \psi(\xi) \mathcal{L}_T \phi(\xi) d\xi = \int \phi(\xi) \mathcal{L}_T^* \psi(\xi) d\xi,$$

(1.84)

for rapidly decaying smooth test functions $\phi$ and $\psi$. Integration by parts yields

$$\mathcal{L}_T^* = \frac{1}{2} \frac{\partial^2}{\partial \xi^2} (\sigma^2(T, \xi) \cdot) - \frac{\partial}{\partial \xi} (\mu(T, \xi) \cdot),$$

(1.85)

where the rapid decay of the test functions ensures that the boundary terms are zero. Let us define $\tilde{p}(T, \xi)$ as the solution of the equation

$$\frac{\partial \tilde{p}}{\partial T} = \mathcal{L}_T^* \tilde{p},$$

(1.86)

with the initial condition $\tilde{p}(t, \xi) = \delta(\xi - x)$ at $T = t$. Now, for $t \leq s \leq T$, compute

$$\frac{d}{ds} \int \tilde{p}(s, \xi') u(s, \xi') d\xi' = \int (u(s, \xi') \mathcal{L}_s^* \tilde{p}(s, \xi') - \tilde{p}(s, \xi') \mathcal{L}_s u(s, \xi')) d\xi' = 0,$$

where we have used the equations (1.82) and (1.86) for $u$ and $\tilde{p}$, and the defining property (1.84) of the adjoint. Consequently

$$\int \tilde{p}(t, \xi') u(t, \xi') d\xi' = \int \tilde{p}(T, \xi') u(T, \xi') d\xi',$$

and using the initial condition for $\tilde{p}$ and the terminal condition for $u$, we obtain

$$u(t, x) = \int \tilde{p}(T, \xi') h(\xi') d\xi',$$
for an arbitrary function $h$. By (1.83), we identify $\bar{p}(T, \xi) = p(t, x; T, \xi)$. Therefore, the transition probability $p(t, x; T, \xi)$ satisfies the forward Kolmogorov equation, also known as Fokker–Planck equation:

$$\frac{\partial p}{\partial T} = \mathcal{L}_p^* p, \quad T > t,$$

$$p(t, x; t, \xi) = \delta(\xi - x).$$

The rigorous derivation of these Kolmogorov equations can be found in the references given in the Notes at the end of the chapter.

### 1.5.4 Application to the Black–Scholes Partial Differential Equation

In the previous section, we have assumed the existence, uniqueness, and regularity of the solution of the partial differential equation (1.80) in order to apply Itô’s formula. A sufficient condition for this is that the coefficients $\mu$ and $\sigma$ are regular enough and that the operator $\mathcal{L}_t$ is uniformly elliptic, meaning (in this one-dimensional situation) that there exists a positive constant $A$ such that

$$\sigma^2(t, x) \geq A > 0 \quad \text{for every } t \geq 0 \text{ and } x \in \mathcal{D},$$

so that the diffusion coefficient $\sigma(t, x)$ cannot become too small. Here $\mathcal{D}$ is the domain of the process $(X_t)$, which may be natural (e.g., $\mathcal{D} = \{x > 0\}$ for the geometric Brownian motion) or imposed externally from other modeling considerations.

When $\mu(t, x) = rx$ and $\sigma(t, x) = \sigma x$ in (1.80), we have the Black–Scholes partial differential equation (1.40) for the option price $P(t, x)$ on the domain $\{x > 0\}$, since

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \mathcal{L} - r,$$

where $\mathcal{L}$ is the infinitesimal generator of the geometric Brownian motion $X$. The ellipticity condition (1.88) is clearly not satisfied since the diffusion coefficient $\sigma x$ goes to zero as the state variable approaches zero. We get around this difficulty here (and also in more general situations) with the change of variable $P(t, x) = u(t, y = \log x)$, so that equation (1.40) becomes

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial y^2} + \left( r \left( 1 - \frac{1}{2} \sigma^2 \right) \frac{\partial u}{\partial y} - ru \right) = 0,$$

(1.89)

to be solved for $0 \leq t \leq T$, $y \in \mathcal{Y}$ and with the final condition $u(T, y) = h(e^y)$. The operator

$$\mathcal{L} = \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial y^2} + \left( r \left( 1 - \frac{1}{2} \sigma^2 \right) \frac{\partial}{\partial y} \right),$$

is the infinitesimal generator of the (nonstandard) Brownian motion defined by

$$dY_t = \left( r \left( 1 - \frac{1}{2} \sigma^2 \right) \right) dt + \sigma dW^*_t,$$

where $(W^*_t)$ is a standard Brownian motion under $\mathbb{P}^*$. We use here the same notation as in the equivalent martingale measure context, but the only important fact is that $W^*$ is a standard Brownian motion with respect to the probability used to compute the expectation in the Feynman–Kac formula (1.81). Applying this formula to $Y_t$ yields

$$u(t, y) = \mathbb{E}^* \left\{ e^{-r(T-t)} h \left( e^{y+(r-\frac{1}{2} \sigma^2)(T-t))+\sigma(W^*_T-W^*_t)} \right) \mid Y_t = y \right\},$$

which is indeed the same as (1.69) by undoing the change of variable $e^y = x$. 

34
1.6 American Options and Free-boundary Problems

We recall here basic facts about American options under constant volatility which will be used in Chapter 7 where we derive corrections due to stochastic volatility.

1.6.1 Optimal Stopping

Pricing American derivatives is mathematically more involved than the European case. Using the theory of optimal stopping, it can be shown that the price of an American derivative with payoff function $h$ is obtained by maximizing over all the stopping times the expected value of the discounted payoff. As in the European case, the expectations have to be taken with respect to the risk-neutral probability to avoid arbitrage opportunities. In other words, the intuitive idea presented in Section 1.2.2 is correct when $\mathbb{E}$ is replaced by $\mathbb{E}^*$:

$$P(0, x) = \sup_{\tau \leq T} \mathbb{E}^* \left\{ e^{-r\tau} h(X_\tau) \right\},$$

is the price of the derivative at time $t = 0$, when $X_0 = x$ and where the supremum is taken over all the possible stopping times less that the expiration date $T$. This formula can be generalized to get the price of American derivatives at any time $t$ before expiration $T$:

$$P(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left\{ e^{-r(\tau-t)} h(X_{t, x}^\tau) \right\},$$

(1.90)

where $(X_{s, x}^t)_{s \geq t}$ is, as in Section 1.5, the stock price starting at time $t$ from the observed price $x$.

By taking $\tau = t$ we deduce that $P(t, x) \geq h(x)$, which is natural since if $P(t, x) < h(x)$ then there would be an obvious instant arbitrage at time $t$. Moreover by choosing $t = T$ we obtain $P(T, x) = h(x)$.

Because an American derivative gives its holder more rights than the corresponding European derivative, the price of the American is always greater than or equal to the price of the European derivative which has the same payoff function and the same expiration date. By taking $\tau = T$ in (1.90), we see that this is indeed the case.

Formula (1.90) gives the price of an American derivative. The supremum in (1.90) is reached at the optimal stopping time,

$$\tau^* = \tau^*(t) = \inf \left\{ t \leq s \leq T \mid P(s, X_s) = h(X_s) \right\},$$

(1.91)

the first time after $t$ that the price of the derivative drops down to its payoff. In order to determine $\tau^*$, one must first compute the price. In terms of partial differential equations, this leads to a so-called free-boundary value problem. To illustrate, we consider the case of an American put option defined in Section 1.2.2.

It can be shown by a no-arbitrage argument that, for nonnegative interest rates and no dividend paid, the price of an American call option is the same as its corresponding European option. The price of an American put option

$$P^a(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E}^* \left\{ e^{-r(\tau-t)} (K - X_{t, x}^\tau)^+ \right\},$$

is in general strictly higher than the price of the corresponding European put option which has been obtained in closed-form (1.47). In fact, we saw in Figure 1.3 that the Black–Scholes European put option pricing function crosses below the payoff “ramp” function $(K - x)^+$ for small enough $x$. This violates $P(t, x) \geq h(x)$, so the European formula for a put cannot also give the price of the American contract, as is the case for call options. We therefore use a put option as our canonical example of an American security in Chapter 7.
1.6.2 Free-Boundary Value Problems

Pricing functions for American derivatives satisfy partial differential inequalities. For the nonnegative payoff function \( h \), the price of the corresponding American derivative is the solution of the following linear complementarity problem:

\[
P \geq h \\
\mathcal{L}_{BS}(\sigma)P \leq 0 \\
(h - P)\mathcal{L}_{BS}(\sigma)P = 0,
\]

(1.92)

to be solved in \( \{(t, x) : 0 \leq t \leq T, x > 0\} \) with the final condition \( P(T, x) = h(x) \), and where \( \mathcal{L}_{BS}(\sigma) \) is the Black–Scholes operator given by (1.41). The second inequality is linked to the supermartingale property of \( e^{-rt}P(t, X_t) \) through (1.79) applied to \( g = P \).

To see that the price (1.90) is the solution of the differential inequalities (1.92), with the optimal stopping time given by (1.91), we assume that we can apply Itô’s formula to the solution \( P \) of (1.92). For any stopping time \( t \leq \tau \leq T \) we have

\[
e^{-rt}P(\tau, X^t_\tau) = e^{-rt}P(t, x) + \int_t^\tau e^{-rs} \left( \frac{\partial}{\partial t} + \mathcal{L} - r \right) P(s, X^t_s)ds \\
+ \int_t^\tau e^{-rs} \sigma X^t_s \frac{\partial P}{\partial x}(s, X^t_s)dW^*_s,
\]

where \( \mathcal{L} \) is the infinitesimal generator of \( X \). The integrand of the Riemann integral is nonpositive by (1.92) and, since \( \tau \) is bounded, the expectation of the martingale term is zero by Doob’s optional stopping theorem. This leads to

\[
\mathbb{E}^* \left\{ e^{-r(\tau - t)}P(\tau, X^t_\tau) \right\} \leq P(t, x),
\]

and, using the first inequality in (1.92),

\[
\mathbb{E}^* \left\{ e^{-r(\tau - t)}h(X^t_\tau) \right\} \leq P(t, x).
\]

It is easy to see now that if \( \tau = \tau^* \), the optimal stopping time defined in (1.91), then we have equalities throughout. This verifies that if (1.92) has a solution to which Itô’s formula can be applied then it is the American derivative price (1.90).

In the case of the American put option there is an increasing function \( x^*(t) \) – the free boundary – such that, at time \( t \),

\[
P(t, x) = K - x \quad \text{for } x < x^*(t) \\
\mathcal{L}_{BS}(\sigma)P = 0 \quad \text{for } x > x^*(t),
\]

(1.93)

with

\[
P(T, x) = (K - x)^+, \\
x^*(T) = K.
\]

(1.94)

(1.95)

In addition, \( P \) and \( \frac{\partial P}{\partial x} \) are continuous across the boundary \( x^*(t) \), so that

\[
P(t, x^*(t)) = K - x^*(t),
\]

(1.96)

\[
\frac{\partial P}{\partial x}(t, x^*(t)) = -1.
\]

(1.97)
The exercise boundary $x^*(t)$ separates the hold region, where the option is not exercised, from the exercise region, where it is; this is illustrated in Figure 1.4. In the corresponding Figure 1.5, we show the trajectory of the stock price and the optimal exercise time $\tau^*$.

As in (1.89), the change of variable $y = \log x$ is convenient for analytical and numerical purposes. Notice that this is a system of equations and boundary conditions for $P(t, x)$ and the free boundary $x^*(t)$.

### 1.7 Path-Dependent Derivatives

In order to price path-dependent derivatives, one has to compute the expectations of their discounted payoffs with respect to the risk-neutral probability. To illustrate this we give examples which will be developed in the context of stochastic volatility in Chapter 6.

#### 1.7.1 Barrier Options

A down-and-out call option (European style) is an example of a barrier option that has a payoff function given by (1.27). Its no-arbitrage price at time $t = 0$ for a stock price equal to $x$ is given by

$$P(0, x) = \mathbb{E}^* \left\{ e^{-rT} (X_T - K)^+ 1_{\{\inf_0^{t\leq T} X_t > B\}} \mid X_0 = x \right\}.$$ 

The price at time $t < T$ of this option is given by

$$P_t = \mathbb{E}^* \left\{ e^{-r(T-t)} (X_T - K)^+ 1_{\{\inf_0^{t\leq T} X_t > B\}} \mid \mathcal{F}_t \right\}$$

$$= 1_{\{\inf_0^{t\leq T} X_s > B\}} \mathbb{E}^* \left\{ e^{-r(T-t)} (X_T - K)^+ 1_{\{\inf_t^{T\leq T} X_s > B\}} \mid \mathcal{F}_t \right\}$$

$$= 1_{\{\inf_0^{t\leq T} X_s > B\}} u(t, X_t),$$
where we have used the Markov property of \((X_t)\), and \(u(t, x)\) is defined by

\[
 u(t, x) = \mathbb{E}^* \left\{ e^{-r(T-t)} (X_T - K)^+ \mathbf{1}_{\{\inf_{t \leq s \leq T} X_s > B\}} \mid X_t = x \right\}.
\]  

(1.98)

These expectations can be computed by using classical results on the joint probability distribution of the Brownian motion and its minimum, obtained by the reflection principal. This involves to work with the non-standard Brownian motion \(\log X_t\), and then perform a change of measure, by Girsanov’s theorem, in order to remove the drift (we refer to (Etheridge 2002) for instance). Alternatively, the function \(u(t, x)\) given by (1.98) satisfies the following boundary value problem on \(\{x > B\}\):

\[
\mathcal{L}_{BS}(\sigma)u = 0, \\
u(t, B) = 0, \\
u(T, x) = (x - K)^+.
\]

We briefly explain here how the method of images leads to a formula for \(u(t, x)\) in terms of the Black–Scholes price of the corresponding call option. As in the probabilistic approach mentioned above, we first use log-coordinates, \(y = \log x\), by defining

\[
v(t, y) = u(t, e^y),
\]

which satisfies the following problem on \(\{y > \log B\}\):

\[
\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial y^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial v}{\partial y} - ry = 0, \\
v(t, \log B) = 0, \\
v(T, y) = (e^y - K)^+.
\]
We then reduce the problem to the heat equation by defining

\[ k = \frac{2r}{\sigma^2}, \]

(1.99)

and introducing

\[ w(t, y) = e^{\left[ \frac{1}{2}(k-1)y - \frac{\sigma^2}{2}(k+1)^2 t \right]} v(t, y), \]

which satisfies

\[ \frac{\partial w}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial y^2} = 0, \]

\[ w(t, \log B) = 0, \]

\[ w(T, y) = e^{\left[ \frac{1}{2}(k-1)y - \frac{\sigma^2}{2}(k+1)^2 T \right]} (e^y - K)^+, \]

on the domain \( \{ y > \log B \} \). We now consider the solution \( w_1(t, y) \) of the heat equation

\[ \frac{\partial w_1}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 w_1}{\partial y^2} = 0, \]

on the full domain \( \{-\infty < y < +\infty\} \) with the terminal condition

\[ w_1(T, y) = e^{\left[ \frac{1}{2}(k-1)2\log B - \frac{\sigma^2}{2}(k+1)^2 T \right]} (e^{2\log B - y} - K)^+ \text{ if } y > \log B, \text{ and 0 elsewhere.} \]

One can easily check that the image function \( w_2(t, y) = w_1(t, 2\log B - y) \) satisfies the same equation on the full space with the terminal condition

\[ w_2(T, y) = e^{\left[ \frac{1}{2}(k-1)(2\log B - y) - \frac{\sigma^2}{2}(k+1)^2 T \right]} (e^{2\log B - y} - K)^+ \text{ if } y < \log B, \text{ and 0 elsewhere.} \]

Consequently, the function \( w_1(t, y) - w_2(t, y) \) satisfies the heat equation, its value at the boundary \( \{ y = \log B \} \) is \( w_1(t, \log B) - w_2(t, \log B) = 0 \), and it has the same terminal condition than \( w \) on \( \{ y > \log B \} \). Therefore its restriction to the domain \( \{ y > \log B \} \) is equal to \( w \), and one has

\[ w(t, y) = w_1(t, y) - w_1(t, 2\log B - y). \]

Undoing the previous changes of variable, we get

\[ u(t, x) = v(t, \log x) = e^{\left[ \frac{1}{2}(k-1)\log x - \frac{\sigma^2}{2}(k+1)^2 t \right]} w(t, \log x) \]

\[ = e^{\left[ \frac{1}{2}(k-1)\log x - \frac{\sigma^2}{2}(k+1)^2 t \right]} (w_1(t, \log x) - w_1(t, 2\log B - \log x)) \]

\[ = u_{BS}(t, x) - (\frac{x}{B})^{1-k} u_{BS} \left( t, \frac{B^2}{x} \right), \]

(1.100)

where \( u_{BS}(t, x) \) is the Black–Scholes price of the European derivative with payoff function \( h(x) = (x - K)^+1_{[x>B]} \). In the case \( B \leq K \), where the knock-out barrier is below the call strike, \( u_{BS}(t, x) \) is simply the price \( C_{BS}(t, x) \) of a call option given by the Black–Scholes formula (1.42).
1.7.2 Lookback Options

We consider for instance a floating strike lookback put which pays the difference of the realized maximum of the underlying asset during the option’s life and the asset price itself at the expiration time $T$. Defining the running maximum $J_t = \sup_{0 \leq s \leq t} X_s$, its payoff is $J_T - X_T$, and its value at time $t = 0$, for $X_0 = x$, is given by

$$P(0, x) = IE^* \{ e^{-rT} (J_T - X_T) \mid X_0 = x \} = xe^{-rT} IE^* \{ \sup_{0 \leq t \leq T} (e^{(r-\frac{1}{2}\sigma^2)t}\sigma W_t) \} - x,$$

by using the martingale property of the discounted stock price under the risk neutral probability $\mathbb{P}^*$, and the explicit form of $X_t$. By using log-variables and a change of measure, the computation is reduced to integrals involving the joint distribution of a driftless Brownian motion and its running maximum. Alternatively, we briefly present the partial differential equation approach which will be used in Chapter 6.

The price $P(t, x, J)$ of this option satisfies the problem

$$\mathcal{L}_{BS}(\sigma)P = 0 \quad \text{in} \quad x < J \text{ and } t < T,$$

$$\frac{\partial P}{\partial J}(t, J, J) = 0,$$

$$P(T, x, J) = J - x.$$

The boundary condition at $J = x$ expresses the fact that the price of the lookback option for $X_t = J_t$ is insensitive to small changes in $J_t$ because the realized maximum at time $T$ is larger than the realized maximum at time $t$ with probability one.

The problem of finding $P(t, x, J)$ can be reduced to a one (space) dimensional boundary value problem with the following similarity reduction:

$$\xi = x/J, \quad \text{and} \quad P(t, x, J) = JQ(t, \xi).$$

We can express $Q(t, \xi)$ as the solution of

$$\mathcal{L}_{BS}(\sigma)Q = 0 \quad \text{for} \quad \xi < 1 \text{ and } t < T,$$

$$\left( \frac{\partial Q}{\partial \xi} - Q \right)(t, 1) = 0,$$

$$Q(T, \xi) = 1 - \xi,$$

where, in a slight abuse of notation, we redefine $\mathcal{L}_{BS}(\sigma)$ as the Black–Scholes operator with respect to the variable $\xi$. We now use the log-variable

$$\eta = \log \xi, \quad u(t, \eta) = Q(t, \xi),$$

and find that $u(t, \eta)$ satisfies the problem

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial \eta^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u}{\partial \eta} - ru = 0 \quad \text{in} \eta < 0 \text{ and } t < T,$$

$$\left( \frac{\partial u}{\partial \eta} - u \right)(t, 0) = 0,$$

$$u(T, \eta) = 1 - e^\eta.$$
We first find \( w(t, \eta) = \frac{\partial u}{\partial \eta}(t, \eta) - u(t, \eta) \) which solves the following (Dirichlet) boundary value problem

\[
\frac{\partial w}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial \eta^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial w}{\partial \eta} - r w = 0 \quad \text{in } \eta < 0 \text{ and } t < T,
\]

with the conditions \( w(t, 0) = 0 \) and \( w(T, \eta) = -1 \). The solution for \( w(t, \eta) \) can be found via the method of images explained in Section 1.7.1:

\[
w(t, \eta) = e^{-r(T-t)} \left[ e^{(1-k)\eta} N(c_1(T-t)) - N(c_2(T-t)) \right],
\]

where

\[
c_1(\tau) = \frac{\eta}{\sigma \sqrt{\tau}} + \frac{1}{2} (1 - k) \sigma \sqrt{\tau} \quad \text{and} \quad c_2(\tau) = \frac{-\eta}{\sigma \sqrt{\tau}} + \frac{1}{2} (1 - k) \sigma \sqrt{\tau},
\]

and \( k \) was defined in (1.99). Restoring all transformations and using the notation of (Wilmott, Howison and Dewynne 1996), we get

\[
P(t, x, J) = -x + x \left( 1 + \frac{\sigma^2}{2r} \right) N(d_1) + Je^{-r(T-t)} \left( N(d_5) - \frac{\sigma^2}{2r} \left( \frac{x}{J} \right)^{1-k} N(d_6) \right),
\]

where

\[
d_5 = \frac{\log(J/x) - (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_6 = \frac{\log(x/J) - (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}},
\]

\[
d_7 = \frac{\log(x/J) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}.
\]

### 1.7.3 Forward-Start Options

Recall from Section 1.2.3 that a typical forward-start option is a call option maturing at time \( T \) such that the strike price is set equal to \( X_{T_1} \) at time \( T_1 < T \). Its payoff at maturity \( T \) is given by \( h = (X_T - X_{T_1})^+ \). If \( T_1 \leq t \leq T \), the contract is simply a call option with (known) strike price \( K = X_{T_1} \); its price \( P(t, x) \) is given by the Black–Scholes formula (1.42) where \( X_t = x \). When \( t < T_1 < T_2 \), which is the case when the contract is initiated, its price at time \( t \) is given by \( P(t, X_t) \) where

\[
P(t, x) = \mathbb{E}^* \left\{ e^{-r(T-t)}(X_T - X_{T_1})^+ \mid X_t = x \right\}
\]

\[
= \mathbb{E}^* \left\{ e^{-r(T_1-t)} \mathbb{E}^* \left\{ e^{-r(T-T_1)} (X_T - X_{T_1})^+ \mid \mathcal{F}_{T_1} \right\} \mid X_t = x \right\}
\]

\[
= \mathbb{E}^* \left\{ e^{-r(T_1-t)} C_{BS}(T_1, X_{T_1}; T, K = X_{T_1}) \mid X_t = x \right\}
\]

\[
= \mathbb{E}^* \left\{ e^{-r(T_1-t)} X_{T_1} \left( N(d_1) - e^{-r(T-T_1)} N(d_2) \right) \mid X_t = x \right\},
\]

where \( d_1 \) and \( d_2 \) are given here by

\[
d_1 = \left( r + \frac{1}{2} \sigma^2 \right) \frac{\sqrt{T-T_1}}{\sigma}, \quad d_2 = \left( r - \frac{1}{2} \sigma^2 \right) \frac{\sqrt{T-T_1}}{\sigma},
\]

41
because the underlying call option is computed at the money \( K = X_{T_1} \). We then deduce

\[
P(t, x) = \left( N(\bar{d}_1) - e^{-r(T-T_1)}N(\bar{d}_2) \right) \mathbb{E}^* \left\{ e^{-r(T_1-t)}X_{T_1} \mid X_t = x \right\}
\]

\[
= x \left( N(\bar{d}_1) - e^{-r(T-T_1)}N(\bar{d}_2) \right),
\]

by using the martingale property of the discounted stock price under the risk neutral probability \( \mathbb{P}^* \).

### 1.7.4 Compound Options

We consider the example of a call-on-call option introduced in Section 1.2.3. For \( t < T_1 < T \), at time \( T_1 \), the maturity time of the option, the payoff is given by

\[
h(C_{BS}(T_1, X_{T_1}; K, T)) = (C_{BS}(T_1, X_{T_1}; K, T) - K_1)^+,
\]

where \( C_{BS}(T_1, X_{T_1}; K, T) \) is the price at time \( T_1 \) of a call option with strike \( K \) and maturity \( T \). The price at time \( t \) of this call-on-call is given by

\[
P(t, x) = \mathbb{E}^* \left\{ e^{-r(T_1-t)} (C_{BS}(T_1, X_{T_1}; K, T) - K_1)^+ \mid X_t = x \right\}
\]

\[
= \mathbb{E}^* \left\{ e^{-r(T_1-t)} (C_{BS}(T_1, X_{T_1}; K, T) - K_1) 1_{\{X_{T_1} \geq x_1\}} \mid X_t = x \right\},
\]

where \( x_1 \) is defined by \( C_{BS}(T_1, x_1; K, T) = K_1 \). We then write

\[
P(t, x) = \mathbb{E}^* \left\{ e^{-r(T_1-t)} (C_{BS}(T_1, X_{T_1}; K, T) - K_1)^+ 1_{\{X_{T_1} \geq x_1\}} \mid X_t = x \right\}
\]

\[
= e^{-r(T_1-t)} \mathbb{E}_{t,x}^* \left\{ X_T 1_{\{X_T \geq K\}} 1_{\{X_{T_1} \geq x_1\}} \right\} - K e^{-r(T_1-t)} \mathbb{E}_{t,x}^* \left\{ 1_{\{X_T \geq K\}} 1_{\{X_{T_1} \geq x_1\}} \right\}
\]

\[
- K e^{-r(T_1-t)} \mathbb{E}_{t,x}^* \left\{ 1_{\{X_{T_1} \geq x_1\}} \right\}
\]

\[
= x N_2 (d_1^{(1)}, d_1; \rho) - K e^{-r(T_1-t)} N_2 (d_2^{(1)}, d_2; \rho) - K_1 e^{-r(T_1-t)} N (d_2^{(1)}),
\]

where we used the following transformations and notations:

\[
X_{T_1} = x \exp \left( \left( r - \frac{1}{2} \sigma^2 \right)(T_1 - t) + \sigma \sqrt{T_1 - t} Z_1 \right), \quad Z_1 = \frac{W_{T_1} - W_t}{\sqrt{T_1 - t}}
\]

\[
X_T = x \exp \left( \left( r - \frac{1}{2} \sigma^2 \right)(T - t) + \sigma \sqrt{T - t} Z \right), \quad Z = \frac{W_T - W_t}{\sqrt{T - t}}
\]

\[
d_1^{(1)} = \frac{\log(x/x_1) + (r + \frac{1}{2} \sigma^2)(T_1 - t)}{\sigma \sqrt{T_1 - t}}, \quad d_2^{(1)} = d_1^{(1)} - \sigma \sqrt{T_1 - t}
\]

\[
d_1 = \frac{\log(x/K) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}
\]

\[
N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy, \quad \rho = \sqrt{T_1 - t}/\sqrt{T - t}
\]

\[
N_2(z_1, z; \rho) = \frac{1}{2\pi} \int_{-\infty}^{z_1} \int_{-\infty}^{z} e^{-(y_1^2 + 2\rho y_1 y + y^2)/2} dy_1 dy
\]

42
1.7.5 Asian Options

As an example we consider an **Asian (European style) average-strike option** whose payoff is given by a function of the stock price at maturity and of the arithmetically-averaged stock price before maturity like in an average strike call option (1.32). Without entering into further details, one can introduce the integral process

\[ I_t = \int_0^t X_s ds, \]

and redo the replicating strategies analysis or the risk-neutral valuation argument for the pair of processes \((X_t, I_t)\). Observe that \((I_t)\) does not introduce new risk or, in other words, there is no new Brownian motion in the equation \(dI_t = X_t dt\). Using a two-dimensional version of Itô’s formula presented in the Section 1.9, one can deduce the partial differential equation

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + r \left( x \frac{\partial P}{\partial x} - P \right) + x \frac{\partial P}{\partial I} = 0,
\]

(1.104)

to be solved, for instance, with the final condition \(P(T, x, I) = (x - \frac{I}{T})^+\), in order to obtain the price \(P(t, X_t, I_t)\) of an arithmetic-average strike call option at time \(t\). This is solved numerically in most examples. Dimension reduction techniques have been proposed (see for instance (Vecer 2002)).

1.8 First Passage Structural Approach to Default

In this section, in the context of credit risk, we present Merton’s approach to default as an illustration of the use of the objects introduced in this chapter. We consider here the problem of pricing a defaultable zero-coupon bond which pays a fixed amount (say \(S\)) at maturity \(T\) unless default occurs, in which case it is worth nothing. In other words, we consider the simple case of no recovery in case of default.

1.8.1 Merton’s Approach

In Merton’s approach, the underlying \(X_t\) follows a geometric Brownian motion, and default occurs if \(X_T < B\) for some threshold value \(B\). In this case the price at time \(t\) of the defaultable bond is simply the price of a European digital option which pays one if \(X_T\) exceeds the threshold and zero otherwise, as in (1.70). Assuming that the underlying is tradable and the risk free interest rate \(r\) is constant, by no-arbitrage argument, the price of this option is explicitly given by \(u^d(t, X_t)\) where

\[
\begin{align*}
u^d(t, x) & = E^* \left\{ e^{-r(T-t)} 1_{X_T > B} | X_t = x \right\} \\
& = e^{-r(T-t)} P^* \{ X_T > B | X_t = x \} \\
& = e^{-r(T-t)} P^* \left\{ \left( r - \frac{\sigma^2}{2} \right) (T-t) + \sigma (W_t^* - W_T^*) > \log \left( \frac{B}{x} \right) \right\} \\
& = e^{-r(T-t)} P^* \left\{ \frac{W_T^* - W_t^*}{\sqrt{T-t}} > -\frac{\log \left( \frac{B}{x} \right) + \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}} \right\} \\
& = e^{-r\tau} N(d_2(\tau)),
\end{align*}
\]
with the usual notation \( \tau = T - t \) and the distance to default:

\[
d_2(\tau) = \frac{\log \left( \frac{\tau}{T} \right) + \left( r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.
\]  

(1.106)

1.8.2 First Passage Model

In the first passage structural approach, default occurs if \( X_t \) goes below \( B \) at some time before maturity. In this extended Merton, or Black and Cox model, the payoff is

\[
h(X) = 1_{\{\inf_{0 \leq s \leq T} X_s > B\}}.
\]

The defaultable bond can then be viewed as a path-dependent derivative. Its value at time \( t \leq T \), denoted by \( P^B(t, T) \), is given by

\[
P^B(t, T) = \mathbb{E}^* \left\{ e^{-r(T-t)}1_{\{\inf_{0 \leq s \leq T} X_s > B\}} \mid \mathcal{F}_t \right\} = 1_{\{\inf_{0 \leq s \leq t} X_s > B\}} e^{-r(T-t)} \mathbb{E}^* \left\{ 1_{\{\inf_{t \leq s \leq T} X_s > B\}} \mid \mathcal{F}_t \right\}.
\]

Indeed \( P^B(t, T) = 0 \) if the asset price has reached \( B \) before time \( t \), which is reflected by the factor \( 1_{\{\inf_{0 \leq s \leq t} X_s > B\}} \). Introducing the default time \( \tau_t \) defined by

\[
\tau_t = \inf\{s \geq t, X_s \leq B\},
\]

one has

\[
\mathbb{E}^* \left\{ 1_{\{\inf_{t \leq s \leq T} X_s > B\}} \mid \mathcal{F}_t \right\} = \mathbb{P}^* \{ \tau_t > T \mid \mathcal{F}_t \},
\]

which shows that the problem reduces to the characterization of the distribution of default times. Observe that \( \tau_t \) is a stopping time (as defined in Section 1.2.2). Moreover, it is a predictable stopping time in the sense that there exists an increasing sequence of stopping times \( \tau_t^{(n)} < \tau_t \) such that \( \lim_{n \to \infty} \tau_t^{(n)} = \tau \) a.s. For instance one can consider the sequence \( \{\tau_t^{(n)}\} \) defined by

\[
\tau_t^{(n)} = \inf\{s \geq t, X_s \leq B + 1/n\}.
\]

These stopping times are illustrated in Figure 1.6.

An alternative intensity based approach to default, presented in Chapter 14, consists of introducing default times which are not predictable.

In the first passage model, a defaultable zero-coupon bond is in fact a binary down-an-out barrier option where the barrier level and the strike price coincide. As presented in Section 1.7.1, we have

\[
\mathbb{E}^* \left\{ 1_{\{\inf_{t \leq s \leq T} X_s > B\}} \mid \mathcal{F}_t \right\} = \mathbb{P}^* \left\{ \inf_{t \leq s \leq T} \left( r - \frac{\sigma^2}{2} \right) (s - t) + \sigma (W_s^* - W_t^*) \right\} > \log \left( \frac{B}{x} \right) \mid X_t = x, \]

which can be computed by using the distribution of the minimum of a (non standard) Brownian motion. In this Markovian setting, we have

\[
\mathbb{E}^* \left\{ e^{-r(T-t)}1_{\{\inf_{t \leq s \leq T} X_s > B\}} \mid \mathcal{F}_t \right\} = u(t, X_t),
\]

where \( u(t, x) \) is the solution of the following partial differential equations problem:

\[
\begin{align*}
L_{BS}(\sigma) u &= 0 \quad \text{on} \quad x > B, \; t < T, \\
u(t, B) &= 0 \quad \text{for any} \quad t \leq T, \\
u(T, x) &= 1 \quad \text{for} \quad x > B,
\end{align*}
\]

(1.108)
which is to be solved for \( x > B \). This problem can be solved by introducing the corresponding European digital option which pays $1 at maturity if \( X_T > B \) and nothing otherwise. Its price at time \( t < T \) is given by \( u^d(t, X_t) \), where \( u^d(t, x) \) is computed explicitly in (1.105). The function \( u^d(t, x) \) is the solution to the partial differential equation

\[
\mathcal{L}_{BS}(\sigma) u^d = 0 \quad \text{on} \quad x > 0, \quad t < T,
\]

\[
u^d(T, x) = 1 \quad \text{for} \quad x > B, \quad \text{and} \quad 0 \quad \text{otherwise}.
\]

By using the method of images presented in Section 1.7.1, the solution \( u(t, x) \) of the problem (1.108) can be written

\[
u(t, x) = u^d(t, x) - \left( \frac{x}{B} \right)^{1-k} u^d\left( t, \frac{B^2}{x} \right),
\]

where \( k \) was defined in (1.99). Combining the expression (1.105) for \( u^d(t, x) \) with (1.110), we get

\[
u(t, x) = e^{-r(T-t)} \left( N(d_2^+(T-t)) - \left( \frac{x}{B} \right)^{1-k} N(d_2^-(T-t)) \right),
\]

where we denote

\[
d_2^\pm(\tau) = \frac{\pm \log \left( \frac{x}{B} \right) + \left( r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.
\]

This simple model of default will be further discussed and generalized in Chapter 13.

1.9 Multidimensional Stochastic Calculus

In the following chapters we consider models where volatility is driven by additional processes (or factors). In order to handle these situations, we will need multidimensional versions of the tools introduced in the previous sections. We briefly summarize here the multidimensional Itô formula, Girsanov’s Theorem, and the Feynman–Kac formula.
1.9.1 Multi-dimensional Itô’s Formula

We consider the generalization of the stochastic differential equations (1.11) to the case of systems of such equations:

\[ dX^i_t = \mu_i(t, X_t)dt + \sum_{j=1}^d \sigma_{i,j}(t, X_t)dW^j_t, \quad i = 1, \cdots, d, \tag{1.113} \]

where \( W^j_t, \quad j = 1, \cdots, d, \) are \( d \) independent standard Brownian motions, and

\[ X_t = (X^1_t, \cdots, X^d_t), \]

is a \( d \)-dimensional process. We assume that the functions \( \mu_i(t, x) \) and \( \sigma_{i,j}(t, x) \) are smooth and at most linearly growing at infinity, so that this system has a unique solution adapted to the filtration \( (\mathcal{F}_t) \) generated by the Brownian motions \( (W^j_t) \). We now consider real processes of the form \( f(t, X_t) \) where the real function \( f(t, x) \) is smooth on \( \mathbb{R}_+ \times \mathbb{R}^d \) (for instance continuously differentiable with respect to \( t \) and twice continuously differentiable in the \( x \)-variable). The \( d \)-dimensional Itô formula can then be written:

\[ df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^d \frac{\partial f}{\partial x^i}(t, X_t)dx^i_t + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x^i \partial x^j}(t, X_t)d\langle X^i, X^j \rangle_t, \tag{1.114} \]

where

\[ d\langle X^i, X^j \rangle_t = \sum_{k=1}^d \sigma_{ik}(t, X_t)\sigma_{jk}(t, X_t)dt \]

\[ = (\sigma \sigma^T)_{i,j}(t, X_t)dt, \tag{1.115} \]

follows from the cross-variation rules \( d\langle t, W^j_t \rangle = d\langle W^i_t, W^j_t \rangle = 0 \) for \( i \neq j \), and \( d\langle W^i_t, W^i_t \rangle = dt \). Formula (1.114) can then be rewritten:

\[ df(t, X_t) = \left( \frac{\partial f}{\partial t} + \sum_{i=1}^d \mu_i \frac{\partial f}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^T)_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dt + \sum_{i=1}^d \frac{\partial f}{\partial x^i} \left( \sum_{j=1}^d \sigma_{i,j}dW^j_t \right), \tag{1.116} \]

where the partial derivatives of \( f \) and the coefficients \( \mu \) and \( \sigma \) are evaluated at \( (t, X_t) \).

1.9.2 Girsanov’s Theorem

In Section 1.4.1 we have used a change of probability measure so that the one-dimensional process \( W^*_t = W_t + \theta t \) becomes a standard Brownian motion under the new probability \( \mathbb{P}^\theta \). We now give a multidimensional version of this result in the case where \( \theta \) may also be a stochastic process. To simplify the presentation we assume that the \( d \)-dimensional process \( (\theta_j(X_t), j = 1, \cdots, d) \) where the functions \( \theta_j(x) \) are bounded (for less restrictive conditions, such as Novikov condition, we refer to (Karatzas and Shreve 1991)). Generalizing (1.58), we define the real process \( (\xi^\theta_t)_{0 \leq t \leq T} \) by:

\[ \xi^\theta_t = \exp \left( -\sum_{j=1}^d \left( \int_0^t \theta_j(X_s)dW^j_s + \frac{1}{2} \int_0^t \theta_j^2(X_s)ds \right) \right), \tag{1.117} \]
which satisfies

\[ d\xi_t^0 = -\xi_t^0 \sum_{j=1}^d \theta_j(X_t) dW_t^j. \]

and, therefore, is a martingale. We then define, on $\mathcal{F}_T$, the probability $\mathbb{P}^*$ by $d\mathbb{P}^* = \xi_T^0 d\mathbb{P}$ as in (1.59). Girsanov’s theorem states that the processes $(W_t^j)_{0 \leq t \leq T}, j = 1, \ldots, d$, defined by

\[ W_t^j = W_0^j + \int_0^t \theta_j(X_s) ds, \quad j = 1, \ldots, d, \quad (1.118) \]

are independent standard Brownian motions under $\mathbb{P}^*$. Using norm and dot product in $\mathbb{R}^d$, the following formal computation, for all $u \in \mathbb{R}^d$, explains this result:

\[
\mathbb{E}^* \left\{ e^{iu \cdot (W_T^* - W_0^*)} \mid \mathcal{F}_s \right\} = \mathbb{E} \left\{ e^{(\xi_T^0 - \xi_s^0) u \cdot (W_T^* - W_0^*)} \mid \mathcal{F}_s \right\}
\]

\[
= \mathbb{E} \left\{ e^{-\frac{1}{2}(t-s)||u||^2} \left[ e^{-su \cdot (W_T^* - W_0^*)} \mid \mathcal{F}_s \right] \right\}
\]

\[
= e^{-\frac{1}{2}(t-s)||u||^2} \mathbb{E} \left\{ e^{-\frac{1}{2}(u-s) \theta \cdot dv} \right\}
\]

\[
= e^{-\frac{1}{2}(t-s)||u||^2} \mathbb{E} \left\{ e^{-(\xi_T^0 - \xi_s^0) u \cdot (W_T^* - W_0^*)} \mid \mathcal{F}_s \right\}
\]

\[
= e^{-\frac{1}{2}(t-s)||u||^2},
\]

where we used the martingale property of $(\xi_t^0 - \xi_u^0)$, and the characterization of independent standard Brownian motions by conditional characteristic functions.

### 1.9.3 The Feynman–Kac Formula

The infinitesimal generator of the (possibly non-homogeneous) Markovian process $X = (X^1, \ldots, X^d)$, introduced in (1.113), is given by

\[
\mathcal{L}_t = \sum_{i=1}^d \mu_i(t, x) \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^T)_{i,j}(t, x) \frac{\partial^2}{\partial x^i \partial x^j}.
\]

If $r(t, x)$ is a function on $\mathbb{R}_+ \times \mathbb{R}^d$ (for instance bounded), then the function $u(t, x)$ defined by

\[
u(t, x) = \mathbb{E} \left\{ e^{-\int_t^T r(s, X_s) ds} h(X_T) \mid X_t = x \right\},
\]

satisfies the partial differential equation

\[
\frac{\partial u}{\partial t} + \mathcal{L}_t u - ru = 0,
\]

with the terminal condition $u(T, x) = h(x)$. Here $h$ does not need to be smooth, and, in particular, can be the payoff of a call, put or digital option.

Such parabolic partial differential equations with an additional source will be important in our perturbation theory: if the function $g(t, x)$ is, for instance, bounded, then the backward problem

\[
\frac{\partial v}{\partial t} + \mathcal{L}_t v - rv + g = 0
\]

\[
v(T, x) = h(x),
\]

admits the solution

\[
v(t, x) = \mathbb{E} \left\{ e^{-\int_t^T r(s, X_s) ds} h(X_T) + \int_t^T e^{-\int_s^T r(u, X_u) du} g(s, X_s) ds \mid X_t = x \right\}.
\]
1.10 Complete Market

The model we have analyzed in this chapter is an example of a complete market model. The simplest definition of a complete market is one in which every contingent claim can be replicated by a self-financing trading strategy in the stock and bond.

In Section 1.3.2, we constructed such a strategy for any European derivative with payoff $h(X_T)$, using the Markov property and Itô’s lemma, and outlined the arguments for American and other exotic contracts. In fact, in this model, any security whose payoff $H$ is known on date $T$ ($H$ is any $\mathcal{F}_T$-measurable random variable with $\mathbb{E}\{H^2\} < \infty$) can be replicated by some unique self-financing trading strategy, as the martingale representation theorem of Section 1.4.3 tells us. Equivalently, such a claim can be perfectly hedged (without overshooting) by trading in the underlying stock and bond. If there is no early exercise feature to the contract (nothing is paid out except on date $T$ and perhaps dividends on fixed dates, or at a fixed rate), then the risk-neutral valuation formula (1.66) prices the derivative.

Finally, we mention that another characterization of an arbitrage-free complete market is that there is a unique equivalent martingale measure $\mathbb{P}^*$ under which the discounted prices of traded securities are martingales. When looking at stochastic volatility market models in the next chapter, we shall see that the market is incomplete: there is a whole family of equivalent martingale measures and derivatives securities cannot be perfectly hedged with just the stock and bond.

1.11 Notes

The original derivation of the no-arbitrage price of a European call option under the lognormal model appeared in (Black and Scholes 1973), with related results in (Merton 1973). The geometric Brownian motion model for the risky asset and many other issues regarding the pricing of options prior to the Black–Scholes theory are discussed in (Samuelson 1973). For further details about the material outlined in Section 1.1, namely Brownian motion, stochastic integrals, stochastic differential equations and Itô’s formulas we recommend (Oksendal 2007) or (Mikosch 1999). These books also cover Girsanov’s and martingale representation theorems discussed in Section 1.4.1 and 1.4.3 as well as the optimal stopping theory introduced in Section 1.6 for the American option pricing problem, and an extensive list of further references on the subject. For the relation with partial differential equations, Kolmogorov’s equations and the Feynman–Kac formula, we refer for instance to (Freidlin 1985) or (Friedman 2006).

There are many books discussing the finance topics we have summarized in this chapter. Among them are (Duffie 2001), (Hull 2008), (Björk 2004), (Lamberton and Lapeyre 1996), (Musiela and Rutkowski 2002), (Etheridge 2002), (Shreve 2004), (Platen and Heath 2006), and (Jeanblanc, Yor and Chesney 2009).

A reference for the method of images approach to pricing barrier options mentioned in Section 1.7 is (Wilmott et al. 1996). Details about the linear complementarity and partial differential inequality formulations of the American pricing problem are found here also. The other examples of exotic options can also be found in the above mentioned references and in (Lipton 2001). We refer to the original work (Goldman, Sosin and Gatto 1979) for lookback options, and to (Geske 1979) for compound options.

For an introduction to Credit Risk we refer for instance to the survey article (Giesecke 2004), and the books (Duffie and Singleton 2003), (Lando 2004) and (Schönbucher 2003).