Dynamic Asset Allocation: 
a Portfolio Decomposition Formula 
and Applications

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1 Introduction

- Dynamic consumption-portfolio choice:
  - Merton (1971): optimal portfolio includes intertemporal hedging terms in addition to mean-variance component (diffusion)
  - Breeden (1979): hedging performed by holding funds giving best protection against fluctuations in state variable (diffusion)
  - Ocone and Karatzas (1991): representation of hedging terms using Malliavin derivatives (Ito, complete markets)
    → Interest rate hedge
    → Market price of risk hedge
  - Detemple, Garcia and Rindisbacher (DGR JF, 2003): practical implementation of model (diffusion, complete markets)
    → Based on Monte Carlo Simulation
    → Flexible method: arbitrary # assets and state variables, non-linear dynamics, arbitrary utility functions
    → Extends to incomplete/frictional markets (DR MF, 2005)
Contribution:

- **New decomposition of optimal portfolio (hedging terms):**
  - Formula rests on change of numéraire: use pure discount bonds as units of account
  - Passage to a new probability measure: forward measure (Geman (1989) and Jamshidian (1989))
  - General context: Ito price processes, general utilities
• New economic insights about structure of hedges:
  → Utility from terminal wealth: hedge
    · fluctuations in instantaneous price of long term bond with maturity date matching investment horizon
    · fluctuations in future bond return volatilities and future market prices of risk (forward density)
    · first hedge has a static flavor (static hedge)
  → Utility from terminal wealth and intermediate consumption
    · static hedge is a coupon-paying bond, with variable coupon payments tailored to consumption needs
  → Risk aversion properties:
    · if risk aversion approaches one both hedges vanish: myopia
    · if risk aversion becomes large mean-variance term and second hedge vanish: holds just long term bonds
    · if risk aversion vanishes all terms are of first order in risk tolerance.
  → Non-Markovian $N + 2$ fund separation theorem.
• Technical contribution:
  → Exponential version of Clark-Haussmann-Ocone formula
  → Identifies volatilities of exponential martingale in terms of Malliavin derivatives
  → Malliavin derivatives of functional SDEs
  → Explicit solution of a Backward Volterra Integral Equation (BVIE) involving Malliavin derivatives.
Applications:
- Preferred habitat
- Extreme risk aversion behavior
- International asset allocation
- Preferences for I-bonds
- Integration of risk management and asset allocation

Road map:
- Model with utility from terminal wealth
- The Ocone-Karatzas formula
- New representation
- Intermediate consumption
- Applications
- Conclusions
2 The Model

- **Standard Continuous Time Model:**
  - Complete markets and Ito price processes
  - Brownian motion $W$, $d$-dimensional
  - Flow of information $\mathcal{F}_t = \sigma(W_s : s \in [0, t])$
  - Finite time period $[0, T]$
  - Possibly non-Markovian dynamics
Assets: Price Evolution

- **Risky assets (dividend-paying assets):**
  \[
  \frac{dS^i_t}{S^i_t} = (r_t - \delta^i_t) \, dt + \sigma^i_t \, (\theta_t \, dt + dW_t), \quad S^i_0 \text{ given}
  \]
  * \(\sigma^i_t\): volatility coefficients of return process \((1 \times d \text{ vector})\)
  * \(r_t\): instantaneous rate of interest
  * \(\delta^i_t\): dividend yield
  * \(\theta_t\): market prices of risk associated with \(W\) \((d \times 1 \text{ vector})\)
  * \((r, \delta, \sigma, \theta)\): progressively measurable processes; standard integrability conditions

- **Riskless asset:**
  * pays interest at rate \(r\)
Investment and Wealth:

- **Portfolio policy** $\pi$: $d$-dimensional, progressively measurable; integrability conditions
  
  $\rightarrow$ amounts invested in assets: $\pi$
  
  $\rightarrow$ amount in money market: $X - \pi'1$

- **Wealth process:**
  \[
  dX_t = r_t X_t dt + \pi'_t \sigma_t (\theta_t dt + dW_t), \text{ subject to } X_0 = x.
  \]

- **Admissibility:** $\pi$ is admissible ($\pi \in \mathcal{A}$) if and only if wealth is non-negative: $X \geq 0$. 
Asset Allocation Problem:

- **Investor maximizes expected utility of terminal wealth:**
  \[
  \max_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T)]
  \]

- **Utility function:** \( U : \mathbb{R}_+ \rightarrow \mathbb{R} \)
  - Strictly increasing, strictly concave and differentiable
  - Inada conditions: \( \lim_{X \to \infty} U'(X) = 0 \) and \( \lim_{X \to 0} U'(X) = \infty \)
- Includes **CRRA** \( U(x) = \frac{1}{1-R} X^{1-R} \) where \( R > 0 \).

- **Property:**
  - Strictly decreasing marginal utility in \((0, \infty)\)
  - Inverse marginal utility \( I(y) \) exists and satisfies \( U'(I(y)) = y \)
  - Derivative: \( I'(y) = 1/U''(I(y)) \)
3 The Optimal Portfolio

- Complete Markets:
  - Market price of risk: \( \theta_t = (\theta_{1t}, ..., \theta_{dt})' \)
  - State price density:
    \[
    \xi_v \equiv \exp \left( - \int_0^v \left( r_s + \frac{1}{2} \theta'_s \theta_s \right) ds - \int_0^v \theta'_s dW_s \right)
    \]
  - Converts state-contingent payoffs into values at date 0
  - Conditional state price density:
    \[
    \xi_{t,v} \equiv \exp \left( - \int_t^v \left( r_s + \frac{1}{2} \theta'_s \theta_s \right) ds - \int_t^v \theta'_s dW_s \right) = \frac{\xi_v}{\xi_t}
    \]

\[
\pi_t^* = \pi^m_t + \pi^r_t + \pi^\theta_t
\]

where

MV:
\[
\pi^m_t = \mathbb{E}_t [\xi_{t,T} \Gamma^*_T] (\sigma'_t)^{-1} \theta_t
\]

IRH:
\[
\pi^r_t = - (\sigma'_t)^{-1} \mathbb{E}_t \left[ \xi_{t,T} (X^*_T - \Gamma^*_T) \int_t^T \mathcal{D}_t r_s ds \right]
\]

MPRH:
\[
\pi^\theta_t = - (\sigma'_t)^{-1} \mathbb{E}_t \left[ \xi_{t,T} (X^*_T - \Gamma^*_T) \int_t^T (dW_s + \theta_s ds)' \mathcal{D}_t \theta_s \right]
\]

- Optimal terminal wealth \( X^*_T = I(y^* \xi_T) \)
- Constant \( y^* \) solves \( x = E [\xi_T I(y^* \xi_T)] \) (static budget constraint)
- \( \Gamma(X) \equiv -U'(X)/U''(X) \): measure of absolute risk tolerance
- \( \Gamma^*_T \equiv \Gamma(X^*_T) \): risk tolerance evaluated at optimal terminal wealth
- \( \mathcal{D}_t \) is Malliavin derivative
Structure of Hedges:

IRH:

$$\pi_t^r = - (\sigma_t')^{-1} E_t \left[ \xi_{t,T} (X_T^* - \Gamma_T^*) \int_t^T D_t r_s ds \right]'$$

- Driven by sensitivities of future IR and MPR to current innovations in $W_t$. Sensitivities measured by Malliavin derivatives $D_t r_s$ and $D_t \theta_s$.

- Sensitivities are adjusted by factor $\xi_{t,T} (X_T^* - \Gamma_T^*)$: depends on preferences, terminal wealth and conditional state prices.

- Optimal terminal wealth: $I(y^* \xi_T)$

- Date $t$ cost: $\xi_{t,T} I(y^* \xi_T) = \xi_{t,T} I(y^* \xi_t \xi_{t,T})$

- Sensitivity to change in conditional SPD $\xi_{t,T}$

$$\frac{\partial (\xi_{t,T} I(y^* \xi_t \xi_{t,T}))}{\partial \xi_{t,T}} = I(y^* \xi_t \xi_{t,T}) + y^* \xi_t \xi_{t,T} I'(y^* \xi_t \xi_{t,T}) = X_T^* - \Gamma_T^*$$

- Sensitivity of conditional SPD to fluctuations in IR and MPR

$$-\xi_{t,T} \int_t^T D_t r_s ds \quad \text{and} \quad -\xi_{t,T} \int_t^T (dW_s + \theta_s ds)' D_t \theta_s.$$
Constant Relative Risk Aversion (CRRA)

\[
\frac{\pi^m_t}{X^*_t} = \frac{1}{R} (\sigma'_t)^{-1} \theta_t
\]

\[
\frac{\pi^f_t}{X^*_t} = -\rho (\sigma'_t)^{-1} \mathbb{E}_t \left[ \frac{\xi^\rho_T}{\mathbb{E}_t[\xi^\rho_T]} \int_t^T D_t r_s ds \right]'
\]

\[
\frac{\pi^\theta_t}{X^*_t} = -\rho (\sigma'_t)^{-1} \mathbb{E}_t \left[ \frac{\xi^\rho_T}{\mathbb{E}_t[\xi^\rho_T]} \int_t^T (dW_s + \theta_s ds)'D_t \theta_s \right]'
\]

- \( \rho = 1 - 1/R \)
- \( y^* = (\mathbb{E} [\xi^\rho_T] / x)^R \)
- \( X^*_t = \mathbb{E}_t [\xi_t,T(y^*\xi_T)^{-1/R}] \)

- Hedging terms are weighted averages of the sensitivities of future interest rates and market prices of risk to the current Brownian innovations.
4 A New Decomposition of the Optimal Portfolio

4.1 Bond Pricing and Forward Measures

- **Pure Discount Bond Price:** \( B_t^T = E_t [\xi_{t,T}] \)

- **Forward \( T \)-Measure:** (Geman (1989) and Jamshidian (1989))
  - **Random variable:**
    \[
    Z_{t,T} \equiv \frac{\xi_{t,T}}{E_t [\xi_{t,T}]} = \frac{\xi_{t,T}}{B_t^T}
    \]
  - **Properties:** \( Z_{t,T} > 0 \) and \( E_t [Z_{t,T}] = 1 \). Use \( Z_{t,T} \) as density
  - **Probability measure:** \( dQ_t^T = Z_{t,T} dP \)
    \[\rightarrow\] Equivalent to \( P \)
Change of Numéraire: unit of account is $T$-maturity bond

- Under $Q^T_t$ price $V(t)$ of a contingent claim with payoff $Y_T$ is

$$V(t) = E_t [\xi_{t,T} Y_T] = E_t [\xi_{t,T}] E_t \left[ \frac{\xi_{t,T}}{E_t [\xi_{t,T}]} Y_T \right] = B^T_t E_t [Y_T]$$

- $E^T_t [\cdot] \equiv E_t [Z_{t,T} \cdot]$ is expectation under $Q^T_t$

- **Martingale property:** $V(t) / B^T_t = E_t [Y_T] = E_t [Z_{t,T} Y_T]$.

- Density $Z_{t,T}$ is **stochastic discount factor:** converts future payoffs into current values measured in bond unit of account.
Characterization (Theorem 2): The forward $T$-density is given by

$$Z_{t,T} \equiv \exp \left( \int_t^T \sigma^Z(s,T)' \, dW_s - \frac{1}{2} \int_t^T \sigma^Z(s,T)' \sigma^Z(s,T) \, ds \right)$$

- **volatility at** $s \in [t,T]$: $\sigma^Z(s,T) \equiv \sigma^B(s,T) - \theta_s$
- **bond return volatility**: $\sigma^B(s,T)' \equiv \mathcal{D}_s \log B^T_s$

Contribution(s):

- **Identify volatility of forward measure**
- **Application of Exponential Clark-Haussmann-Ocone formula**
- **Market price of risk in the numéraire**
4.2 Portfolio allocation and long term bonds

▶ An Alternative Portfolio Decomposition Formula:

\[ \pi^*_t = \pi^m_t + \pi^b_t + \pi^z_t \]

- **Mean variance demand:**

\[ \pi^m_t = E_t^T \left[ \Gamma^*_T \right] B_t^T (\sigma'_t)^{-1} \theta_t \]

- **Hedge** motivated by fluctuations in price of **pure discount bond with matching maturity**

\[ \pi^b_t = (\sigma'_t)^{-1} \sigma^B (t, T) E_t^T \left[ X^*_T - \Gamma^*_T \right] B_t^T \]

- **Hedge** motivated by fluctuations in **density of forward T-measure**

\[ \pi^z_t = (\sigma'_t)^{-1} E_t^T \left[ (X^*_T - \Gamma^*_T) D_t \log(Z_{t,T}) \right]' B_t^T \]
Essence of Formula: change of numéraire

- **SPD representation:** \( \xi_{t,T} = B_t^T Z_{t,T} \)
- **Optimal terminal wealth:** \( X^*_T = I (y^* \xi_t B_t^T Z_{t,T}) \)
- **Cost of optimal terminal wealth:** \( B_t^T Z_{t,T} I (y^* \xi_t B_t^T Z_{t,T}) \)
- **Hedging portfolio:** \( D_t (B_t^T Z_{t,T} I (y^* \xi_t B_t^T Z_{t,T})) \)
- **Chain rule of Malliavin calculus:**

\[
\begin{align*}
&\rightarrow (Z_{t,T} I (y^* \xi_t B_t^T Z_{t,T}) + B_t^T Z_{t,T} I' (y^* \xi_t B_t^T Z_{t,T}) y^* \xi_t Z_{t,T}) D_t B_t^T \\
&\rightarrow (B_t^T I (y^* \xi_t B_t^T Z_{t,T}) + B_t^T Z_{t,T} I' (y^* \xi_t B_t^T Z_{t,T}) y^* \xi_t B_t^T) D_t Z_{t,T} \\
&\rightarrow B_t^T Z_{t,T} I' (y^* \xi_t B_t^T Z_{t,T}) B_t^T Z_{t,T} D_t (y^* \xi_t)
\end{align*}
\]


- **Long Term Bond Hedge:**
  - Immunizes against instantaneous fluctuations in return of long term bond with matching maturity date
  - Corresponds to portfolio that maximizes the correlation with long term bond return
  - This portfolio is a synthetic asset or maturity matching bond itself, if exists

- **Forward Density Hedge:**
  - Immunizes against fluctuations in forward density $Z_{t,T}$ (instantaneous and delayed)
  - Source of fluctuations are bond return volatilities and MPRs:
    \[
    \sigma^Z (s, T) \equiv \sigma^B (s, T) - \theta_s
    \]
  - $\mathcal{D}_t \sigma^Z (s, T) = \mathcal{D}_t \sigma^B (s, T) - \mathcal{D}_t \theta_s$. 
Remarks:

- Generality of decomposition is remarkable:
  - Interest rate’s response to Brownian innovations has disappeared
  - Replaced by bond volatilities and MPRs
  - Surprising because infinite dim. Itô processes:
    - Model for prices is not diffusion
    - Current bond prices are not sufficient statistics for IR evolution

- Formula in spirit of immunization strategies sometimes advocated by practitioners
  - First term is static hedge: hedge against current fluctuations in LT bond price
  - To first approximation optimal portfolio has mean-variance term + static hedge
  - Additional hedge fine tunes allocation: captures fluctuations in future quantities
  - Static hedge is preference independent
Signing the Static Hedge:

- Bond prices negatively related to IR
- IR innovation negatively related to equity innovation
- In one factor (BMP) model $\sigma^B > 0$: boost demand for stocks
4.3 Constant Relative Risk Aversion

Hedging Terms are:

\[
\frac{\pi_t^b}{X_t^*} = \rho (\sigma'_t)^{-1} \sigma^B (t, T) B_t^T
\]

\[
\frac{\pi_t^z}{X_t^*} = \rho (\sigma'_t)^{-1} E_t^{T} \left[ \frac{Z_{t,T}^{\rho-1}}{E_t^{T} [Z_{t,T}^{\rho-1}]} D_t \log (Z_{t,T}) \right]' B_t^T
\]

Highlights knife-edge property of log utility (Breeden (1979))

- Logarithmic investor displays myopia (hedging demands vanish)
- More (less) risk averse investors will hold (short) portfolio synthesizing long term bond
- More (less) risk averse investors will hold (short) portfolio that hedges forward density
  - portfolio is individual-specific: depends on risk aversion of utility function
Literature: special cases of this result analyzed by

- Lioui and Poncet (2001) and Lioui (2005):
  - Diffusion models with power utility.
  - Lioui and Poncet (2001): last hedging component in terms of unknown volatility function (PDE).
  - Lioui (2005): affine model with mean-reverting IR and MPR processes. Forward density hedge is proportional to vector of volatilities of MPR with proportionality factor linear in MPRs.
Illustration: optimal stock-bond mix for CRRA investor

- Model:
  - $T$-maturity bond is traded
  - Two assets: Stock and investment horizon matching bond

$$
\sigma_t = \begin{bmatrix}
\sigma_{1t}^{stock} & \sigma_{2t}^{stock} \\
\sigma_{1t}^B & \sigma_{2t}^B
\end{bmatrix}
$$

- Optimal portfolio weight: static hedging component

$$
\frac{\pi_t^b}{X_t^*} = \rho (\sigma'_t)^{-1} \sigma_t^B = \rho \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

- If $\pi_t^z \approx 0$ hedging is very simple:
  - no hedging component for stocks
  - hedging component does not depend on investment horizon
  - hedging portfolio only depends on relative risk aversion coefficient
  - no need to estimate: if risk aversion is $R = 4$, then static hedging component for bonds is 0.75.
Illustration: Asset Allocation Puzzle

- Asset Allocation Puzzle (see Canner, Mankiw and Weil (1997)): investment advisors typically recommend an increase in the bonds-to-equities ratio for more conservative investors while mean-variance portfolio theory predicts that a constant ratio is optimal.

- Bonds-to-equities ratio in Gaussian terms structure models:

\[
e(t, T) = \sigma_t^S \frac{\theta_{2t} + \sigma_{2t}^B (Q(t, T) - 1)}{\sigma_{2t}^B \theta_1 t - \sigma_{1t}^B \theta_2 t}
\]

\[
Q(t, T) \equiv E_t^T [X_T^*] / E_t^T [\Gamma_T^*]
\]

- For HARA utility \( u(x) = (x - A)^{1 - R}/(1 - R) \),

\[
* \quad Q(t, T; R) \equiv R \left( 1 + \left( \frac{B_0^T h(0, t; R)}{x/A - B_0^T} \right) \left( \frac{B_0^T / B_0^T}{B_t^T Z_t} \right)^{-1/R} \right) = R \left( 1 + \frac{A B_t^T}{X_t^* - A B_t^T} \left( \frac{B_0^T / B_0^T}{B_0^T} \right) \right)
\]

* with

\[
* \quad h(0, t; R) \equiv \exp \left( \frac{(\rho / R)}{\int_0^t \left( \frac{1}{2} \| s + \sigma^B(s, T) \|^2 - \| \sigma^B(s, T) \|^2 \right) ds \right).
\]

- Bonds-to-equities ratio risk tolerance, at a given time \( t \), if and only if \( Q(t, T) \) is a monotone function of risk tolerance.
- In the presence of wealth effect the bonds-to-equities ratio is not necessarily monotone in risk aversion

- **Vasicek interest rate model:** \( r_0 = \bar{r} = 0.06, \; \kappa_r = 0.05, \; \sigma_{r_1} = -0.02, \; \sigma_{r_2} = -0.015 \) and market prices of risk are constants \( \theta_s = 0.3 \) and \( \theta_b = 0.15 \). The interest rate at \( t = 5 \) is \( r_t = 0.02 \).

  Other parameter values are \( A = 200,000, \; x = 100,000 \) and \( T = 10 \).
5 Intermediate Consumption

5.1 The Investor’s Preferences

- **Consumption-portfolio Problem:**
  \[
  \max_{\pi, c \in \mathcal{A}} \mathbb{E} \left[ \int_0^T u(c_t, t) \, dt + U(X_T) \right]
  \]

- **Utility function:** \(u(\cdot, \cdot) : \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R}\) and bequest function: \(U : \mathbb{R}_+ \rightarrow \mathbb{R}\) satisfy standard assumptions

- Maximization over set of admissible portfolio policies \(\pi, c \in \mathcal{A}\)

- Inverse marginal utility function \(J(y, t)\) exists: \(u'(J(y, t), t) = y\) for all \(t \in [0, T]\)

- Inverse marginal bequest function \(I(y)\) exists: \(U'(I(y)) = y\)
5.2 Portfolio Representation and Coupon-paying Bonds

- **Decomposition:**
  \[
  \pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z
  \]

- **Mean variance demand:**
  \[
  \pi_t^m = \left( \int_t^T E_t^v [\Gamma_v^*] B_t^v dv + E_t^T [\Gamma_T^*] B_t^T \right) (\sigma_t')^{-1} \theta_t
  \]

- **Hedge** motivated by fluctuations in price of **coupon-paying bond** with matching maturity:
  \[
  \pi_t^b = (\sigma_t')^{-1} \int_t^T \sigma^B (t, v) B_t^v E_t^v [c_v^* - \Gamma_v^*] dv
  + (\sigma_t')^{-1} \sigma^B (t, T) B_t^T E_t^T [X_T^* - \Gamma_T^*]
  \]

- **Hedge** motivated by fluctuations in **density of forward** \(T\)-measure:
  \[
  \pi_t^z = (\sigma_t')^{-1} \left( \int_t^T E_t^v [(c_v^* - \Gamma_v^*) \mathcal{D}_t \log Z_t,v] B_t^v dv \right)'
  + (\sigma_t')^{-1} \left( E_t^T [(X_T^* - \Gamma_T^*) \mathcal{D}_t \log Z_t,T] B_t^T \right)'
  \]
Static Hedge $\pi^b_t$: hedge against fluctuations in value of coupon-paying bond

- **Coupon payments** $C(v) \equiv E^v_t [c^*_v - \Gamma^*_v]$ at intermediate dates $v \in [0, T)$

- **Bullet payment** $F \equiv E^T_t [X^*_T - \Gamma^*_T]$ at terminal date $T$

- Coupon payments and face value are
  - time-varying
  - tailored to individual’s consumption profile and risk tolerance

- **Bond value**
  
  \[
  B(t, T; C, F) \equiv \int_t^T B^v_t C(v) \, dv + B^T_t F.
  \]

- **Instantaneous volatility**
  
  \[
  \sigma(B(t, T; C, F)) B(t, T; C, F) = \int_t^T \sigma^B(t, v) B^v_t C(v) \, dv + \sigma^B(t, T) B^T_t F
  \]

- **Hedge:** $(\sigma'_t)^{-1} \sigma(B(t, T; C, F)) B(t, T; C, F)$
Forward Density Hedge $\pi^Z_t$:

- Motivation: desire to hedge fluctuations in forward densities $Z_{t,v}$
- Static hedge already neutralizes impact of term structure fluctuations on PV of future consumption
- Given $\xi_{t,v} = B^v_t Z_{t,v}$ it remains to hedge fluctuations in risk-adjusted discount factors $Z_{t,v}, v \in [t, T]$.

Optimal Portfolio Composition:

- To first approximation optimal portfolio has mean-variance term + long term coupon bond hedge
- Under what conditions is this approximation exact (i.e. last term vanishes)?
- If last term does not vanish what is its size?
5.3 Constant Relative Risk Aversion

- Relative risk aversion parameters $R_u, R_U$ for utility and bequest functions. Portfolio:
  
  - Mean-variance term
    \[
    \pi^m_t = (\sigma'_t)^{-1} \left( \int_t^T \frac{1}{R_u} E^v_t [c^*_v] B^v_t dv + \frac{1}{R_U} E^T_t [X^*_T] B^T_t \right) \theta_t
    \]
  
  - Hedge motivated by fluctuations in price of coupon-paying bond with matching maturity
    \[
    \pi^b_t = (\sigma'_t)^{-1} \left( \rho_u \int_t^T \sigma^B (t, v) B^v_t E^v_t [c^*_v] dv + \rho_U \sigma^B (t, T) B^T_t E^T_t [X^*_T] \right)
    \]
  
  - Hedge motivated by fluctuations in densities of forward measures
    \[
    \pi^z_t = \rho_u (\sigma'_t)^{-1} \int_t^T E^v_t [c^*_v \mathcal{D}_t \log Z_{t,v}]' B^v_t dv \\
    + \rho_U (\sigma'_t)^{-1} E^T_t [X^*_T \mathcal{D}_t \log Z_{t,T}]' B^T_t
    \]
Static Hedge has two parts:

- Pure coupon bond (annuity) with coupon given by optimal consumption
- Bullet payment given by optimal terminal wealth
- Two parts are weighted by risk aversion factors $\rho_u$ and $\rho_U$
- Knife edge property traditionally associated with power utility function
- Possibility of positive annuity hedge ($R_u > 1$) combined with negative bequest hedge ($R_U < 1$).
6 Applications

6.1 Preferred Habitats and Portfolio Choice

Preferred Habitat Theory Modigliani and Sutch (1966):

- Individuals exhibit preference for securities with maturities matching their investment horizon
- Investor who cares about terminal wealth should invest in bonds with matching maturity
- Existence of group of investors with common investment horizon might lead to increase in demand for bonds in this maturity range
- Implies increase in bond prices and decrease in yields. Explains hump-shaped yield curves with decreasing profile at long maturities.
Formula shows that optimal behavior naturally induces a demand for certain types of bonds in specific maturity ranges

\[ \pi^*_t = w^m_t (X^*_t - B(t, T; C, F)) + w^b_t B(t, T; C, F) + \pi^z_t \]

\[ w^m_t = \arg \max_w \{ w' \sigma_t \theta_t : w' \sigma_t \sigma_t' w = k \} . \]

\[ w^b_t = \arg \max_w \{ w' \sigma_t \sigma (B(t, T; C)) : w' \sigma_t \sigma_t' w' = k \} \]

\[ \pi^z_t = \arg \max_{\pi} \{ \pi' \sigma_t \hat{\sigma}(t, T) : \pi' \sigma_t \sigma_t' \pi' = k \} \]

where

\[ \hat{\sigma}'_{t,T} \equiv \int_t^T E^v_t [(c^*_v - \Gamma^*_v) D_t \log Z_{t,v}] B^v_t \, dv + E^T_t [(X^*_T - \Gamma^*_T) D_t \log Z_{t,T}] B^T_t \]
• Any individual has preferred bond habitat:
  → Optimal portfolio includes long term bond with maturity date matching the investor’s horizon
  → Preferred instrument is coupon-paying bond with payments tailored to consumption profile of investor

• Complemented by mean-variance efficient portfolio to constitute the static component of allocation

• Under general market conditions the static policy is fine-tuned by dynamic hedge
  → When bond return volatilities and market prices of risk are deterministic, dynamic hedge vanishes
Equilibrium Implications

- Existence of natural preferred habitats in certain segments of fixed income market
- Existence of equilibrium effects on prices and premia in these habitats depends on characteristics of investors’ population
- With sufficient homogeneity
  - Strong demand for structured fixed income products might emerge
  - Prompt financial institutions to offer tailored products appealing to those segments
  - Yields to maturity would then naturally reflect this habitat-motivated demand
Motivation for preferred habitat here is different from Riedel (2001)

- In his model habitat preferences are driven by structure of subjective discount rates placing emphasis on specific future dates
- In our setting preference for long term bonds emerges from the structure of the hedging terms
- Optimal hedging combines static hedge (long term bond) with dynamic hedge motivated by fluctuations in forward measure volatilities
6.2 Universal Fund Separation

- $Y$: vector of $N < d$ state variables with evolution described by the functional stochastic differential equation

$$dY_t = \mu(Y(\cdot))_t dt + \sigma(Y(\cdot))_t dW_t$$

- Suppose that
  - $B^v_t = B(t, v, Y(\cdot))$
  - $\sigma^Z(t, v) = \sigma^Z(t, v, Y(\cdot))$

are Fréchet differentiable functionals of $Y(\cdot)$.

- Universal $N + 1$-fund separation holds: portfolio demands can be synthesized by investing in $N + 2$ (preference free) mutual funds:
  1. riskless asset
  2. the mean-variance efficient portfolio
  3. $N$ portfolios $(\sigma'_t)^{-1} \sigma^Y_t (Y(\cdot))'$ to synthesize the static bond hedge and the forward density hedge.
6.3 Extreme Behavior

• Assume risk tolerances go to zero:

  - Intermediate utility and bequest functions:
    \[(\Gamma_u(z, v), \Gamma_U(z)) \to (0, 0) \text{ for all } z \in [0, +\infty) \text{ and all } v \in [0, T]\]

  - Relative behaviors: for some constant \(k \in [0, +\infty)\):
    \[\frac{\Gamma_u(z_1, v)}{\Gamma_U(z_2)} \to k \text{ for all } z_1, z_2 \in [0, \infty) \text{ and all } v \in [0, T]\]
    \[\frac{\Gamma_u(z_1, v_1)}{\Gamma_u(z_2, v_2)} \to 1 \text{ for all } z_1, z_2 \in [0, \infty) \text{ and all } v_1, v_2 \in [0, T]\]
Limit Allocations: coupon-paying bond with constant coupon $C$ and face value $F$ given by

$$C = \frac{x}{\int_0^T B_0^v dv + B_0^T / k} \quad \text{and} \quad F = \frac{x}{\int_0^T B_0^v dv k + B_0^T}.$$  

- If $k = 0$ exclusive preference for pure discount bond, 
  $$(C, F) = \left(0, \frac{x}{B_0^T}\right)$$
- If $k \to \infty$ preference is for a pure coupon bond, 
  $$(C, F) = \left(\frac{x}{\int_0^T B_0^v dv}, 0\right)$$

Limit Behavior:

- Governed by relation between utility functions at different dates
- As risk tolerances vanish, preference for certainty: coupon-paying bond with bullet payment
- Least extreme of the extreme behaviors drives the habitat:
  - Given a preference for riskless instruments: individuals puts more weight on maturities where risk tolerance is greater
  - Exhibits a time preference in the limit.
Illustration: CARA preferences $\Gamma_u$ and $\Gamma_U$ constant, $k \equiv \Gamma_u / \Gamma_U$.

- Slope of indifference curves:
  \[ -\frac{dX}{dc} = \frac{1}{k} \left( e^{X - c/k} \right)^{\frac{1}{\Gamma_U}} \]
- $k \to 0$
\( k \rightarrow \infty \)
Special case examined by Wachter (2002)

- Arbitrary utility functions over terminal wealth and markets with general coefficients
- Documents emergence of preferred habitat when relative risk aversion goes to infinity
  → Pure discount bond with unit face value and matching maturity
- Our analysis shows that preferred habitat for an extreme consumer may take different forms depending on nature of behavior
  → Pure discount bonds, pure annuities or coupon-paying bonds with bullet payments at maturity can emerge in limit.
Order of Convergence

- As \((\Gamma_u(z, \nu), \Gamma_U(z)) \to (0, 0)\), the limit portfolios
  * \(\bar{\pi}_t^m = \bar{\pi}_t^z = 0\)
  * \(\bar{\pi}_t^b = (\sigma'_t) \int_0^T \sigma^B(t, \nu) B_t^\nu d\nu C + \sigma^B(t, T) B_t^T F\)
- have scaled asymptotic errors:
  * \(\epsilon_t^\alpha(\nu) = (\Gamma_\nu(\cdot))^{-1} (\pi_t^\alpha - \bar{\pi}_t^\alpha)\) with \(\alpha \in \{m, b, z\}\) and \(\nu \in \{u, U\}\),

\[
\begin{align*}
[\epsilon_t^m(U), \epsilon_t^m(u)] & \to (\sigma'_t)^{-1} \theta_t \left[ \int_t^T B_t^\nu d\nu B_t^T \right] \mathcal{K} \\
[\epsilon_t^b(U), \epsilon_t^b(u)] & \to - (\sigma'_t)^{-1} \left[ \int_t^T \sigma^B(t, \nu) B_t^\nu d\nu \sigma^B(t, T) B_t^T \right] \mathcal{K} \\
[\epsilon_t^z(U), \epsilon_t^z(u)] & \to - (\sigma'_t)^{-1} \left[ \int_t^T N_{t,\nu} B_t^\nu d\nu N_{t,T} B_t^T \right] \mathcal{K}
\end{align*}
\]

- where
  * \(N_{t,\tau}\) is given by
    \[
    N_{t,\tau} \equiv E_t^T \left[ \left( \int_t^\tau \sigma^Z(r, \tau)' dW_r - \frac{1}{2} \int_t^\tau \| \sigma^Z(r, \tau) \|^2 dr \right) (D_t \log Z_{t,\tau})' \right]
    \]
  * \(\mathcal{K}\) is given by
    \[
    \mathcal{K} \equiv \begin{bmatrix} k & 1 \\ 1 & \frac{1}{k} \end{bmatrix}
    \]
6.4 Term structure models and asset allocation

Integration of term structure models and asset allocation models:

- Forward rate representation of bonds
  \[
  B_t^v = \exp \left( - \int_t^v f_t^s ds \right)
  \]

  \[\rightarrow\] Continuously compounded forward rate: \( f_t^s \equiv - \frac{\partial}{\partial v} \log (B_t^v) \)

- Bond price volatility:
  \[
  \sigma_B(t, v)' = D_t \log B_t^v = - \int_t^v D_t f_t^s ds = - \int_t^v \sigma_f(t, s) ds
  \]

  \[\rightarrow\] Volatility of forward rate: \( \sigma_f(t, s) \)

- Forward rate dynamics:
  \[
  df_t^v = \sigma_f(t, v) \left( dW_t + (\theta_t - \sigma_B(t, v)) dt \right), \quad f_0^v \text{ given}
  \]

  \[\rightarrow\] Dynamics completely determined by forward rate volatility function and initial forward rate curve
Optimal Portfolio: previous formula with

\[
\mathcal{D}_t \log Z_{t,v} = \int_t^v \left( dW_s + \left( \theta_s + \int_s^v \sigma^f(s,u) \, du \right) \, ds \right)' \left( \mathcal{D}_t \theta_s + \int_s^v \mathcal{D}_t \sigma^f(s,u) \, du \right)
\]

- Forward density hedge in terms of forward rate volatilities
- Useful for financial institution using a specific HJM model to price/hedge fixed income instruments and their derivatives
- Implied forward rates inferred from term structure model and observed prices
  - \( \rightarrow \) estimate volatility function \( \sigma^f(s,u) \)
  - \( \rightarrow \) feed into asset allocation formula
- Simple integration of fixed income management and asset allocation.
Forward Density Hedge:

- Immunization demand due to fluctuations in future market prices of risk and forward rate volatilities.
- Vanishes if deterministic forward rate volatilities $\sigma^f(s,u)$ and market prices of risk $\theta_s$.
- Pure expectation hypothesis holds under forward measure:
  \[ f(t,v) = E^v_t[r_v] \]
  - Standard version of PEH ($f(t,v) = E_t[r_v]$) fails when $Z_{t,v} \neq 1$.
  - Density process $Z_{t,v}$ measures deviation from PEH.
  - Malliavin derivative $D_t \log Z_{t,v}$ captures sensitivity of deviation with respect to shocks.
  - Dynamic hedge = hedge against deviations from PEH.
- If $Z_{t,v} = 1$ PEH holds under the original beliefs and hedging becomes irrelevant.
- If $\sigma^Z$ deterministic, deviations from PEH are non-predictable and do not need to be hedged.
► Literature:

- Extensively employed in practice
- Forward rate volatilities $\sigma^f$ are insensitive to shocks. If MPR also deterministic no need to hedge
- Bajeux-Besnainou, Jordan and Portait (2001) also falls in this category (one factor Vasicek)
Numerical Results: Forward measure hedges in one factor CIR model

- CIR interest rates:

\[ dr_t = \kappa_r (\bar{r} - r_t) dt + \sigma_r \sqrt{r} dW_t; \quad r_0 = r \]

\[ \to \] Parameter values (Durham (JFE, 2003)):

- \( \kappa_r = 0.002 \)
- \( \bar{r} = 0.0497 \)
- \( \sigma_r = -0.0062 \)
- \( r = 0.06 \)

- Market price of risk:

\[ \theta_t = \gamma_r \sqrt{r_t} \]

\[ \to \] Parameter values:

- \( \gamma_r = 0.3 / \sqrt{\bar{r}} \) such that \( \bar{\theta} = \gamma_r \sqrt{\bar{r}} = 0.3 \)

- CRRA preferences for terminal wealth
• Mean-variance demand: \[ \frac{\pi_t^{mv}}{X_t^*} = \frac{1}{R} (\sigma'_t)^{-1} \theta_t \]
Static term structure hedge: \[
\pi_t^b / X_t^* = \rho(\sigma_t')^{-1} \sigma^B(t, T)
\]
- Dynamic forward measure hedge:

\[
\pi_t^*/X_t^* = \rho (\sigma_t')^{-1} E_t^T \left[ \frac{Z_{t,T}^{\rho-1}}{E_t^T [Z_{t,T}^{\rho-1}]} (D_t \log Z_{t,T})' \right]
\]
• Total portfolio weight: \[ \pi_t/X_t^* = \pi_t^{mv}/X_t^* + \pi_t^b/X_t^* + \pi_t^z/X_t^* \]
• Changing initial interest rate: Relative risk aversion fixed at $R = 4$

→ Mean-variance demand: $\frac{\pi_t^{mv}}{X_t^*} = \frac{1}{R} (\sigma'_t)^{-1} \theta_t$
→ Static term structure hedge: \( \frac{\pi_t^b}{X_t^*} = \rho(\sigma_t')^{-1} \sigma^B(t, T) \)
- Dynamic forward measure hedge:

\[
\pi^*_t / X^*_t = \rho (\sigma'_t)^{-1} \mathbb{E}^T_t \left[ \frac{Z^{\rho^{-1}}_t}{\mathbb{E}^T_t[Z^{\rho^{-1}}_{t,T}]} (D_t \log Z_{t,T})' \right]
\]
• Total portfolio weight: 
\[ \pi_t / X_t^* = \pi_t^{m,v} / X_t^* + \pi_t^b / X_t^* + \pi_t^z / X_t^* \]
• Changing initial interest rate: Investment horizon fixed at $T = 15$

→ Mean-variance demand: $\frac{\pi_{t}^{mv}}{X_{t}^{ast}} = \frac{1}{R} (\sigma_{t}')^{-1} \theta_{t}$
→ Static term structure hedge: $\pi_t^b / X_t^* = \rho (\sigma_t')^{-1} \sigma^B(t, T)$
Dynamic forward measure hedge:

\[
\pi_t^z / X_t^z = \rho (\sigma_t')^{-1} E^T_t \left[ \frac{Z_{t,T}^{\rho-1}}{E^T_t[Z_{t,T}^{\rho-1}]} \right] (D_t \log Z_{t,T})'
\]
• Total portfolio weight: 

\[ \pi_t/X^*_t = \frac{\pi^m_t}{X^*_t} + \frac{\pi^b_t}{X^*_t} + \frac{\pi^z_t}{X^*_t} \]
• Approximate forward density portfolio weight:

\[
\pi_{fa}^t / X_t^* = -\sigma_t^{-1} \frac{1}{R_t} E_t^T [N_{t,T}]
\]
- Approximate forward density portfolio weight: \[ \pi^f_{t} / X^*_t = -\sigma_t^{-1} \frac{1}{R} E_t^T \{ N_t, T \} \]
7 Conclusion

▶ Contributions:

- Asset allocation formula based on change of numéraire
- Highlights role of consumption-specific coupon bonds as instruments to hedge fluctuations in opportunity set
- Formula has multiple applications: preferred habitat, extreme behavior, international asset allocation, demand for I-bonds
- Exponential Clark-Haussmann-Ocone formula
- Malliavin derivatives of functional SDEs
- Solution of linear BVIE

▶ Integration of portfolio management and term structure models

- Asset allocation in HJM framework
- Other applications

▶ Universal $N + 2$ fund separation result