# Long-range planning, non-commitment and time-inconsistency

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# Optimal Control

The Ramsey problem.

An infinitely-lived individual or a benevolent government tries to maximize intertemporal welfare

$k(t)$ **capital** at time $t$, $c(t)$ **consumption** at time $t$

$f(k)$ instantaneous **production** if capital is $k$

$f(k(t)) = c(t) + \frac{dk}{dt}$ balance equation

If interest rate is $r > 0$, and **utility of consuming** $c$ is $u(c)$, then the problem at time $t = 0$ is:

$$\max \int_0^\infty e^{-rt} u(c(t))\, dt,$$

$$\frac{dk}{dt} = f(k(t)) - c(t) \text{ and } k(0) = k_0$$

There is a unique *optimal strategy* $c(t) = \sigma(k(t))$ which is given by the Hamilton-Jacobi-Bellman (HJB) equation. In the case when $u(c) = \ln c$, for instance, we have $\sigma(k) = \frac{1}{v'(k)}$ with:

$$\ln v'(k) - f(k) v'(k) = \rho v(k) - 1$$

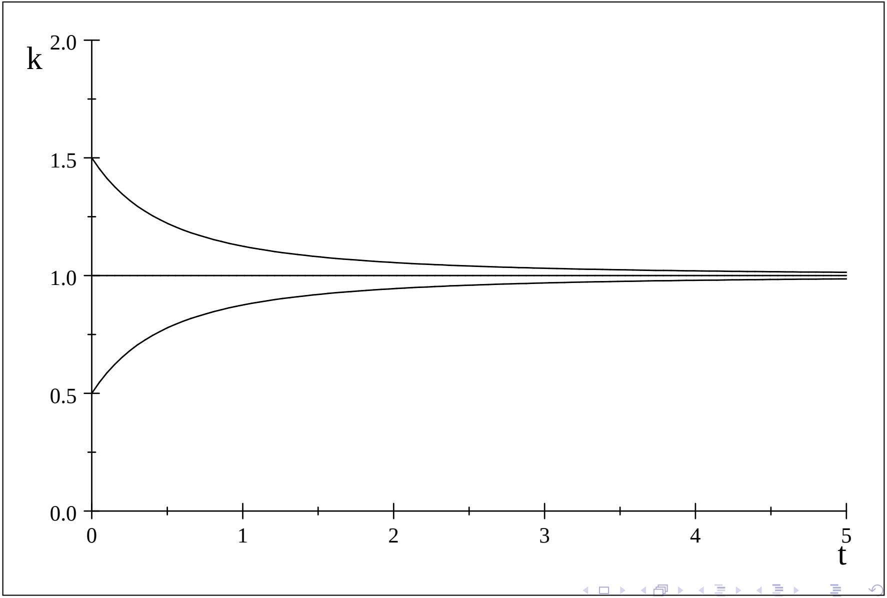
The optimal path converges to a *stationary point*

$$k(t) \longrightarrow k_\infty, \;\; c(t) \longrightarrow c_\infty = f(k_\infty) \text{ when } t \longmapsto \infty$$

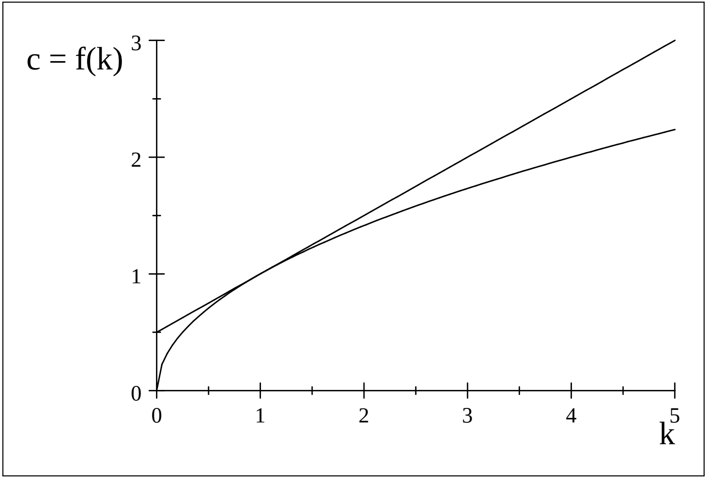
The limit $k_\infty$ is is independent of $k_0$ and $u(c)$. It is characterized by

$$f'(k_\infty) = \rho$$

# Convergence to the stationary state



# The production function and the stationary point




$f'(k) = \rho$

# Non-constant discount rate

Fix some smooth function $r(t)$, and consider the criterion:

$$\max \int_0^\infty e^{-r(t)t} u(c(t))\, dt,$$

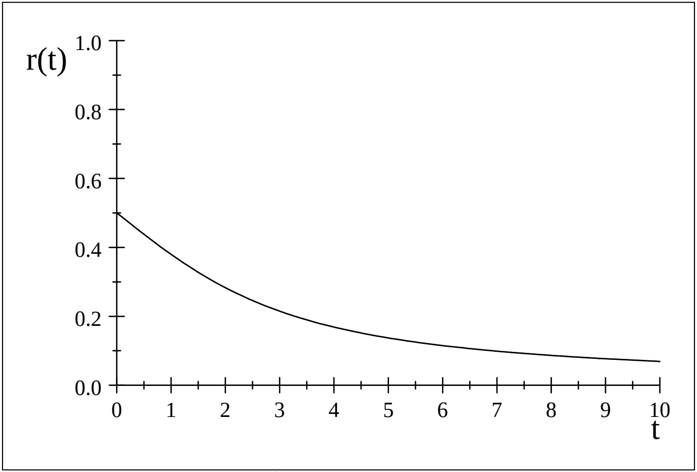
**Example**

Quasi-exponential discount

$$\max \int_0^\infty \left[(1-\lambda) e^{-\rho t} + \lambda e^{-\delta t}\right] u(c(t))\, dt$$

Then $r(t) = -\frac{1}{t} \ln\left[(1-\lambda) e^{-\rho t} + \lambda e^{-\delta t}\right]$

- $r(t) \longrightarrow \lambda\rho + (1-\lambda)\rho$ when $t \longrightarrow 0$
- $r(t) \longrightarrow \min(\rho, \delta)$ when $t \longmapsto \infty$
- $r(t) \longrightarrow \lambda\rho + (1-\lambda)\delta$ when $t \longrightarrow 0$




$\lambda = \frac{1}{2}, \quad \rho = 1, \quad \delta = 0$

# Why non-constant discount rates ?

- There is experimental evidence that individuals are more impatient for the near future than for the distant future. Someone who is indifferent between consuming $c_0$ at some distant time $t$ and $c_1 > c_0$ at time $t+1$ would actually *prefer* consuming $c_0$ right now ($t=0$) than tomorrow ($t=1$)

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- Sumaila and Walters (2005) "*Intergenerational discounting: a new intuitive approach*", Ecological Economics 52, 135-142.

# Intergenerational equity

- the population grows at the rate $\gamma$ and consists of identical individuals
- each generation has a pure rate of time preference $\rho$
- each generation discounts at the rate $\delta > \rho$ the utility of future generations

If everyone consumes a public good $c$ at time $t$, the resulting utility is:

$$u(c)\, e^{-\rho t} \quad \text{for the present generation}$$

$$u(c)\, e^{-\rho(t-s)} \quad \text{for the people born between } s \text{ and } s+ds$$

$$u(c)\left(e^{-\rho t} + \gamma \int_0^t e^{-\delta s} e^{-\rho(t-s)} ds\right) \quad \text{in total}$$

In total:

$$u(c)\left((1-\lambda)\, e^{-\rho t} + \lambda e^{-\delta t}\right) \text{ with } \lambda = \frac{\gamma}{\delta - \rho}$$

# The effect of non-commitment.

Consider an individual who wants to stop smoking:

- If I stop today, I will suffer $-1$ today (withdrawal), but gain $+2$ tomorrow (health)

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- Stopping tomorrow yields a utility of $(-1 + 2\rho)/2 > 0$
- So I decide today to stop tomorrow. Unfortunately, when tomorrow comes, it becomes today, and I decide again to stop the next day !

# Optimality now

Consider the Ramsey problem again

$$\max \int_0^\infty e^{-r(t)} u\left(c\left(t\right)\right) dt, \quad t \geq 0, \ k\left(0\right) = k_0$$

There is a unique optimal control $\bar{c}\left(t\right)$, satisfying the Euler equation:

$$\frac{u''\left(c\left(s\right)\right)}{u'\left(c\left(s\right)\right)} \frac{dc}{ds} + f'\left(k\left(s\right)\right) = r\left(s\right) + sr'\left(s\right) \quad \text{for} \quad 0 \leq s \qquad (0\text{-}optimal)$$

# Optimality tomorrow

Reconsider the problem at $t > 0$

$$\max \int_t^\infty e^{-r(s)} u\left(c\left(s\right)\right) ds, \quad s \geq t, \quad k\left(t\right) = k_t$$

There is a unique optimal trajectory $s \to \left(\tilde{c}\left(s\right), \tilde{k}\left(s\right)\right)$, starting from $\tilde{k}\left(t\right) = \bar{k}\left(t\right)$, satisfying

$$\frac{u''\left(c\left(s\right)\right)}{u'\left(c\left(s\right)\right)} \frac{dc}{ds} + f'\left(k\left(s\right)\right) = r\left(s-t\right) + \left(s-t\right) r'\left(s-t\right) \quad \text{for} \quad t \leq s \tag{$t$-optimal}$$

# Incompatibility

$$\begin{aligned}\frac{u''(c(s))}{u'(c(s))}\frac{dc}{ds}+f'(k(s)) &= r(s)+sr'(s) \quad \text{for } \; 0\leq s \\ \frac{u''(c(s))}{u'(c(s))}\frac{dc}{ds}+f'(k(s)) &= r(s-t)+(s-t)\,r'(s-t) \quad \text{for } \; t\leq s\end{aligned}$$

If $(\bar{c}(t), \bar{k}(t)) = \left(\tilde{c}(s), \tilde{k}(s)\right)$, then this function must satify *both* equations on $s \geq t$. This is clearly not possible unless $r(s)$ is constant. The policy $t \longrightarrow \left(\bar{c}(t), \bar{x}(t)\right)$, which is *optimal* from the time $0$ point of view, is *suboptimal* for all future observers.

# What will replace optimality ?

Optimal control is now useless: any strategy which seems optimal from the perspective of time $t$ will not be implemented at later times $s \geq t$. We will now define a new concept: *equilibrium strategies*

- A strategy $c = \sigma(k)$ has been announced and is public knowledge.

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- She expects all later ones to apply the strategy $\sigma$
- She asks herself if it is in her own interest to apply the same strategy, that is, to consume $\sigma(k(t))$.
- $\sigma$ is an equilibrium strategy if the answer is yes.

# Equilibrium strategies: formal definition

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- On $[t+\varepsilon, \infty]$. the strategy $\sigma$ takes over, and the new trajectory is $k_\varepsilon(t) = k_0(t) + \varepsilon k_1(t)$

$$\frac{dk_1}{ds} = \left(\frac{\partial f}{\partial k}(k_0(s)) - \frac{\partial \sigma}{\partial k}(k_0(s))\right) k_1(s) \tag{1}$$

$$k_1(t+\varepsilon) = \sigma(t,k) - c \quad \text{(first variation)} \tag{2}$$

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- The total gain over $\varepsilon = 0$ is

$$\varepsilon \left[ \begin{array}{c} u(c) - u(\sigma(k(t))) \\ + \int_t^\infty R(s-t) \frac{\partial u}{\partial c}(\sigma(k_0(s))) \frac{\partial \sigma}{\partial k}(s, k_0(s))\, k_1(s)\, ds \end{array} \right]$$

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- We shall say that $\sigma : [0,T] \times R^d \to R^d$ is an *equilibrium strategy* if, for every $t$ and $k$, the maximum in the brackets is attained for $c = \sigma(k(t))$:

# The case of quasi-exponential discount

$$\max \int_0^\infty \left[ (1-\lambda) e^{-\rho t} + \lambda e^{-\delta t} \right] u(c(t))\, dt$$

We shall look for *convergent* equilibrium strategies $c = \sigma(k)$:

$$\frac{dk}{dt} = f(k) - \sigma(k) \Longrightarrow \quad \exists k_\infty : \forall\ k(0),\ k(t) \longrightarrow k_\infty$$

# Necessary conditions

If such an equilibrium strategy $\sigma(k)$ exists, define the *value functions*:

$$\begin{aligned} v(k) &= \int_0^\infty \left(\lambda e^{-\delta t} + (1-\lambda) e^{-\rho t}\right) u(\sigma(k(t)))\, dt \\ w(k) &= \int_0^\infty \left(\lambda e^{-\delta t} - (1-\lambda) e^{-\rho t}\right) u(\sigma(k(t)))\, dt \end{aligned}$$

They satisfy the following system of ODEs:

$$\begin{aligned} \left(f - \frac{1}{v'}\right) v' - \ln v' &= av + bw \\ \left(f - \frac{1}{v'}\right) w' - (2\lambda - 1) \ln v' &= bv + cw \end{aligned}$$

with $a = (\delta + \rho)/2$ and $b = (\delta - \rho)/2$, and we have:

$$\sigma(k) = 1/v'(k), \quad \sigma(k_\infty) = 1/v'(k_\infty) = f(k_\infty)$$

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*If the system*

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*has a* $C^2$ *solution* $v(k), w(k)$ *near* $k_\infty$ *with:*

$$\begin{aligned}v'(k_\infty) &= \frac{1}{f(k_\infty)} \\ av'(k_\infty) + bw'(k_\infty) &= \frac{f'(k_\infty)}{f(k_\infty)}\end{aligned}$$

*then* $\sigma(k) := 1/v'(k)$ *is an equilibrium strategy converging to* $k_\infty$

# An existence theorem

Define $\underline{k} \leq \overline{k}$ by:

$$f'\left(\underline{k}\right) = \lambda\delta + (1-\lambda)\rho, \quad f'\left(\bar{k}\right) = \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}$$

**Theorem**

*For every $k_\infty \in \left[\underline{k}\ ,\ \bar{k}\right]$, there exists an equilibrium strategy converging to $k_\infty$*

$$\frac{dk}{dt} = f(k) - \sigma(k) \Longrightarrow \quad k(t) \longrightarrow k_\infty \quad \forall k(0)$$

For an idea of the proof, see my lectures at PIMS.

- There are *stationary points*, as in the exponential case, but

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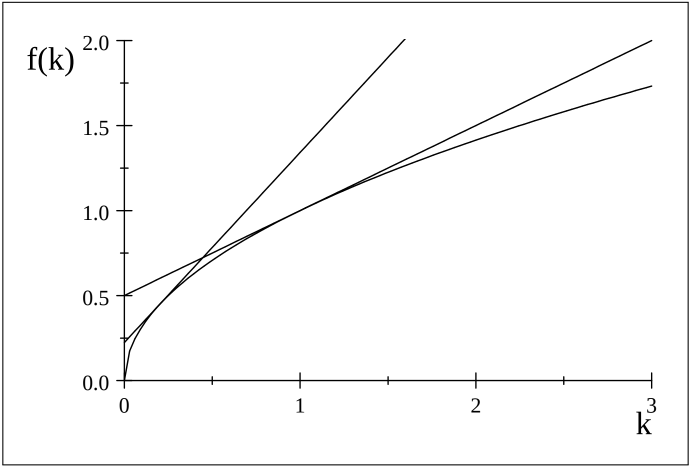
- There are *stationary points*, as in the exponential case, but
- There are *too many* ! The problem of rational choice is not solved yet

# Multiplicity of equilibria




$$0.2 = \underline{k} \leq k_\infty \leq \overline{k} = 1$$

# Stability analysis

Suppose $(v, w)$ converges to $k_\infty$. At a neighbouring point $k$ compare two equilibrium strategies

1. $\sigma_1$ converging to $k_\infty$
2. $\sigma_2$ converging to $k$

The respective benefits starting from $k$ are:

$$\begin{aligned}
v_1 &= \left(\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}\right) \ln f(k_\infty) + v'(k_\infty)(k - k_\infty) \\
v_2 &= \left(\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}\right) \ln f(k) \\
v_2 - v_1 &= \left[\left(\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}\right) f'(k_\infty) - 1\right] \frac{(k - k_\infty)}{f(k_\infty)}
\end{aligned}$$

The second strategy is superior if $k < k_\infty$ which is always possible unless $k_\infty = \underline{k}$

# Conclusion

$$\int_0^\infty [\lambda \exp(-\delta t) + (1-\lambda)\exp(-\rho t)]\, u(c(t))\, dt,$$

$$\frac{dk}{dt} = f(k(t)) - c(t) \text{ and } k(0) = k_0$$

This has infinitely many equilibrium strategies, each of them converging to some $k \in [\underline{k},\ \bar{k}]$

- If $k > \underline{k}$, future generations will eventually find $\underline{k}$ more advantageous

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- The whole strategy then unravels from the end: it is not credible

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- If $k > \underline{k}$, future generations will eventually find $\underline{k}$ more advantageous
- The whole strategy then unravels from the end: it is not credible
- *Rational choice* is $k = \underline{k}$

# Some references

- **For intergenerational equity**:
  Sumaila, U.R., Walters, C., 2005,. "*Intergenerational discounting: a new intuitive approach*", Ecological Economics 52, 135-142.
- **For the general theory in the deterministic case:**
  Ekeland and Lazrak *"Being serious about non-commitment"*

http://arxiv.org/abs/math/0604264

- **For the quasi-exponential case:**

Ekeland and Lazrak, *"Equilibrium policies when preferences are time-inconsistent"*
http://arxiv.org/abs/0808.3790

- **For the stochastic case:**

Ekeland and Pirvu "*Investment and consumption without commitment*", Mathematics and Financial Economics, vol 2 (2008), p. 57-86

# Necessary conditions

If such an equilibrium strategy $\sigma(k)$ exists, define the **value functions**:

$$
\begin{aligned}
v(k) &= \int_0^\infty \left(\theta e^{-r_1 t} + (1-\theta) e^{-r_2 t}\right) u(\sigma(k(t)))\, dt \\
w(k) &= \int_0^\infty \left(\theta e^{-r_1 t} - (1-\theta) e^{-r_2 t}\right) u(\sigma(k(t)))\, dt
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with $a = (r_1 + r_2)/2$ and $b = (r_1 - r_2)/2$, and we have:

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*has a* $C^2$ *solution* $v(k), w(k)$ *near* $k_\infty$ *with:*

$$v'(k_\infty) = \frac{1}{f(k_\infty)}$$
$$av'(k_\infty) + bw'(k_\infty) = \frac{f'(k_\infty)}{f(k_\infty)}$$

*then* $\sigma(k) := 1/v'(k)$ *is an equilibrium strategy converging to* $k_\infty$

# Changing unknowns

$$\mu(k) := av(k) + bw(k) - \ln f(k),$$

Take $(v(k), \mu(k))$ instead of $(v(k), w(k))$ as unknowns. The first equation becomes:

$$fv' - 1 - \ln fv' = \mu$$

- if $\mu < 0$ this equation has no solution

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- if $\mu < 0$ this equation has no solution
- if $\mu = 0$ the only solution is $fv' = 1$
- if $\mu > 0$ it has two solutions $fv' = 1 + x_i(\mu), i = 1, 2$, with $x(0) = 0, \quad x(\mu) \sim \pm\sqrt{\mu}$

# Where the wild things are

$$\begin{aligned}
\frac{d\nu}{dk} &= \frac{1+x(\mu)}{f(k)} \quad \mu(k) \geq 0 \\
\frac{d\mu}{dk} &= \frac{1}{f(k)} \frac{1+x(\mu)}{x(\mu)} D(k,\mu,v) + a\frac{1+x(\mu)}{f(k)} - \frac{f'(k)}{f(k)} \\
D &= a\mu + \left(b^2 - a^2\right) v + a \ln f(k) + (2\theta - 1)\, b \ln \frac{1+x(\mu)}{f(k)}
\end{aligned}$$

- Initial condition:

$$\mu(k_\infty) = 0, \quad v(k_\infty) = \frac{a - (2\theta - 1)\, b}{a^2 - b^2} \ln f(k_\infty), \quad x(0) = 0$$

# Where the wild things are

$$
\begin{aligned}
\frac{dv}{dk} &= \frac{1+x(\mu)}{f(k)} \quad \mu(k) \geq 0 \\
\frac{d\mu}{dk} &= \frac{1}{f(k)} \frac{1+x(\mu)}{x(\mu)} D(k,\mu,v) + a\frac{1+x(\mu)}{f(k)} - \frac{f'(k)}{f(k)} \\
D &= a\mu + \left(b^2 - a^2\right) v + a \ln f(k) + (2\theta - 1)\, b \ln \frac{1+x(\mu)}{f(k)}
\end{aligned}
$$

- Initial condition:

$$
\mu(k_\infty) = 0, \quad v(k_\infty) = \frac{a - (2\theta - 1)\, b}{a^2 - b^2} \ln f(k_\infty), \quad x(0) = 0
$$

- Bearing in mind that $x$ is not smooth $x(\mu) \sim \pm\sqrt{\mu}$

# Changing variables

Take $x$ as the independent variable, so $\mu = x - \ln(1+x)$

$$\frac{dk}{dx} = f(k) \frac{x^2}{1+x} \frac{1}{D(x,k,\nu)},$$
$$\frac{d\nu}{dx} = x^2 \frac{1}{D(x,k,\nu)}$$

- $D(x,k,v) =$
  $\left(2ax + \left(b^2 - a^2\right) v + \left[(2\theta - 1) b - a\right] \ln \frac{1+x}{f}\right)(1+x) - xf$

# Changing variables

Take $x$ as the independent variable, so $\mu = x - \ln(1+x)$

$$\begin{aligned} \frac{dk}{dx} &= f(k)\frac{x^2}{1+x}\frac{1}{D(x,k,\nu)}, \\ \frac{d\nu}{dx} &= x^2\frac{1}{D(x,k,\nu)} \end{aligned}$$

- $D(x,k,v) = \left(2ax + \left(b^2 - a^2\right)v + \left[(2\theta - 1)b - a\right]\ln\frac{1+x}{f}\right)(1+x) - xf$
- Initial condition:

$$x(0) = 0, \quad D(0, k(0), \nu(0)) = 0$$

# Blowup

We introduce a new variable $s$, and the system becomes

$$\begin{aligned}\frac{dx}{ds} &= D(x,k,v)\\ \frac{dk}{ds} &= f(k)\frac{x^2}{1+x}\\ \frac{dv}{ds} &= x^2\end{aligned}$$

The linearized system near $(0, k(0), v(0))$ then is:

$$\frac{d}{dt}\begin{pmatrix} x \\ k \\ v \end{pmatrix} = \begin{pmatrix} a+(2\theta-1)b-f' & (a-(2\theta-1)b)\frac{f'}{f} & b^2-a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ k \\ v \end{pmatrix}$$

# Concluding the proof

If $a+(2\theta-1)\,b \neq f'(k_\infty)$, we can use the central manifold theorem. If $a+(2\theta-1)\,b = f'(k_\infty)$, we must to a further blowup. In all cases, we find a smooth solution(possibly several) with:

$$
\begin{aligned}
v'(k_\infty) &= \frac{1}{f(k_\infty)}, \\
w'(k_\infty) &= \frac{1}{f(k_\infty)}\frac{2f'(k_\infty)-(r_1+r_2)}{r_1-r_2} \\
v''(k_\infty) &= \frac{(r_1-f'(k_\infty))\,(f'(k_\infty)-r_1)}{f'(k_\infty)-\theta r_1-(1-\theta)\,r_2}\frac{1}{f^2(k_\infty)}
\end{aligned}
$$

# More about the degenerate case

Computing $D(x,k,v)$ when $f'(k_\infty) = a + (2\theta - 1)b$, we get:

$$\begin{aligned}
\frac{dx}{ds} &= \alpha(x)x^2 + \beta(k)(k-k_\infty) + \gamma(k)(k-k_\infty)x + \left(b^2 - a^2\right)(1+x)(
\\
\frac{dk}{ds} &= x^2 \frac{f(k)}{1+x}
\\
\frac{dv}{ds} &= x^2
\end{aligned}$$

where $\alpha, \beta, \gamma$ are smooth functions of one variable such that:

$$2\alpha(0) = 3a + (2\theta-1)b,\ \beta(0) = ((2\theta-1)b - a)\frac{f'(k_\infty)}{f(k_\infty)},\ \gamma(0) = \beta(0) -$$

We then perform the change of variables $K(s) := (k(s) - k_\infty)x(s)^{-2}$ and $V(s) := (v(s) - v_\infty)x(s)^{-2}$ One finds $\frac{dK}{ds}(0) = f(\bar{k}) \neq 0$, one can take $K$ instead of $s$ as the independent variable, and we get a regular system of ODEs for $x(K)$ and $V(K)$

# Estimates

**Linearizing** the equation of motion $\frac{dk}{dt} = f(k) - \frac{1}{v'}$ :

$$\frac{dx}{dt} = \left( f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} \right) x$$

Convergence to $k_\infty$ requires that $f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} \leq 0$, hence:

$$\theta r_1 + (1-\theta) r_2 \leq f'(k_\infty) \leq \frac{1}{\frac{\theta}{r_1} + \frac{1-\theta}{r_2}}$$