Volatility and Liquidity Trading

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Outline

The Liquidity Nature of the Financial Markets

- Liquidity Risk
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.
- Market Risk
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The crucial role of understanding the Liquidity Nature of a financial market, for both market participants and regulators alike

- Black Monday in 1987
- LTCM and sovereign bond crisis in 1998
- Riot of subprime credit products in 2007
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Intension/activity of buying and selling does affect the market price of an asset

Trading tactics inspired by the liquidity rationale and trend of the market

- optimal execution for a single player
  Bertsimas & Lo, 98; Almgren & Chriss, 01; Almgren 03
- strategic play between multiple players
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A quick overview of previous models

- Trading in continuous-time $\longrightarrow$ differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role

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$X(t) = \int_0^t \xi(s)\,\mathrm{d}s$, where $\xi(t)$ is the trading intensity in continuous-time.

market (mid-quote) price

$$\mathrm{d}P(t) = \mu(t, \cdots)\mathrm{d}t + f(\xi(t))\mathrm{d}t + \sigma(t, \cdots)\mathrm{d}W(t) \qquad (1)$$

$$COST = \int_0^t \tilde{P}(s)\xi(s)\,\mathrm{d}s = \int_0^t (P(s) + g(\xi(s)))\xi(s)\,\mathrm{d}s$$

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Call a trading scheme a *clean-hand trade* if the strategy $\{\xi(t)\}_{t\in[0,T]}$ satisfies $0 = \int_0^T \xi(t)\,\mathrm{d}t = X(T) - X(0)$, and the integral process $X(t)$ is bounded.

$$\Pi = \mathbb{E}\left[-\int_0^T \tilde{P}(t)\xi(t)\,\mathrm{d}t\right] \leq 0 \tag{2}$$

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\Pi &= \mathbb{E}\left[-\int_0^T (P(t)+g(\xi(t)))\xi(t)\,\mathrm{d}t\right] \qquad (3)\\
&= \mathbb{E}\left[-\int_0^T P(t)\,\mathrm{d}X(t) - \int_0^T g(\xi(t))\xi(t)\,\mathrm{d}t\right]\\
&= \mathbb{E}\left[- P(t)X(t)\big|_0^T + \int_0^T X(t)\left(f(\xi(t))\mathrm{d}t + \sigma(t,P(t))\mathrm{d}W(t)\right)\right.\\
&\qquad \left. - \int_0^T g(\xi(t))\xi(t)\,\mathrm{d}t\right]\\
&= \int_0^T X(t)f(\xi(t))\,\mathrm{d}t - \int_0^T g(\xi(t))\xi(t)\,\mathrm{d}t\\
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$$\begin{aligned}\Pi &= \int_0^{\frac{T}{2}} \xi t f(\xi)\,\mathrm{d}t + \int_{\frac{T}{2}}^{T} \xi (T-t) f(-\xi)\,\mathrm{d}t \\ &\quad - \int_0^{\frac{T}{2}} g(\xi)\xi\,\mathrm{d}t - \int_{\frac{T}{2}}^{T} g(-\xi)(-\xi)\,\mathrm{d}t \\ &= \frac{T^2}{8}\xi\left(f(\xi)+f(-\xi)\right) + \frac{T}{2}\xi\left(g(-\xi)-g(\xi)\right) \\ &\leq 0\end{aligned} \tag{4}$$

$\Rightarrow f(-\xi) = -f(\xi)$, for any $\xi \in \mathbb{R}$
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&\quad - \int_0^{\frac{\xi_2}{\xi_1+\xi_2}T} g(\xi_1)\xi_1\,\mathrm{d}t - \int_{\frac{\xi_2}{\xi_1+\xi_2}T}^{T} g(-\xi_2)(-\xi_2)\,\mathrm{d}t \\
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&\quad -g(\xi_1)\xi_1\frac{\xi_2T}{\xi_1+\xi_2} + g(-\xi_2)\xi_2\frac{\xi_1T}{\xi_1+\xi_2} \\
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- This analytical argument is supported by empirical studies such as Almgren et al. (05) conducted on large-scale datasets traded at NYSE.
- Time-homogeneity can be obtained by rescaling real time using the intra-day volume up to that moment, so-called *volume time*.
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The Story of a Liquidity Trading Game

- A distressed trader with maximum inventory x_0, constrained by an exogenous time horizon $[0, T]$
  - Objective, to generate cash as much as possible
  - Only allowed to monotonely sell (liquidate), not allowed to buy back at any moment during $[0, T]$
- A perfectly solvent trader, sophisticated and aggressive
  - Able to buy or sell at any moment
  - The only constraint is to be *clean-hand* by $\bar{T} \gg T$.
- Each player looks to closed-loop optimal control strategies, aiming to utilize the updates/feedback of market evolution to refine her control.
  - $\Rightarrow$ Subgame-Perfect.

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Setting up the Liquidity Trading Game Model

An illiquid asset with permanent component coef γ, temporary component coef λ, intra-day volatility σ, in common knowledge to the two players

$$\mathrm{d}Z(t) = (\gamma\xi(t) + \gamma\eta(t))\mathrm{d}t + \sigma \mathrm{d}W(t)$$

CL control strategies for the two players

$$\phi(t, x, y, z) \quad \text{and} \quad \psi(t, x, y, z)$$

Given the CL strategy of the opponent, each player solves her optimal control problem

Agreeing at the Nash-equilibrium of this game, when nobody has incentive to deviate.

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The Stochastic Optimal Control Problem

Given the CL strategy $\psi(\cdots)$ of the 2nd player, the optimal control problem for the 1st player

$$U(t,x,y,z) = \min_{\xi(\cdot)\in\mathcal{A}} \mathbb{E}\left[\int_t^T (Z(s) + \lambda(\xi(s) + \psi(s,X(s),Y(s),Z(s)))) \cdot\xi(s)\,\mathrm{d}s \,\middle|\, \begin{matrix} X(t)=x \\ Y(t)=y \\ Z(t)=z \end{matrix}\right]$$

where

$$\begin{aligned} \mathrm{d}Z(t) &= (\gamma\xi(t) + \gamma\psi(t,X(t),Y(t),Z(t)))\mathrm{d}t + \sigma\mathrm{d}W(t) \\ \mathrm{d}X(t) &= \xi(t)\mathrm{d}t \\ \mathrm{d}Y(t) &= \psi(t,X(t),Y(t),Z(t))\mathrm{d}t \end{aligned}$$

The Stochastic Optimal Control Problem

The HJB equation for player 1

$$U_t + \min\left\{\lambda\xi^2 + \lambda\psi(t,x,y,z)\xi + z\xi + \xi U_x + \psi(t,x,y,z)U_y \right.$$

$$\left. +(\gamma\xi + \gamma\psi(t,x,y,z))U_z + \frac{1}{2}\sigma^2 U_{zz} \,|\xi \le 0\right\} = 0$$

$$\begin{aligned} -U_t &= \psi(t,x,y,z)(U_y + \gamma U_z) + \frac{1}{2}\sigma^2 U_{zz} \\ &- \frac{1}{4\lambda}[(z + \lambda\psi(t,x,y,z) + U_x + \gamma U_z)_+]^2 \end{aligned}$$

for $t \in [0,T]$, $x \in [0,x_0]$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, and where

$$\phi(t,x,y,z) = -\frac{1}{2\lambda}\left(z + \lambda\psi(t,x,y,z) + U_x + \gamma U_z\right)_+$$

The Stochastic Optimal Control Problem

Given the CL strategy $\phi(\cdots)$ of the 1st player, the optimal control problem for the 2nd player

$$V(t,x,y,z) = \min_{\eta(\cdot)\in\mathcal{A}} \mathbb{E}\left[\int_t^T (Z(s) + \lambda(\phi(s,X(s),Y(s),Z(s)) + \eta(s))) \cdot\eta(s)\,\mathrm{d}s \,\middle|\, \begin{array}{c} X(t)=x \\ Y(t)=y \\ Z(t)=z \end{array}\right]$$

where

$$\begin{aligned} \mathrm{d}Z(t) &= (\gamma\phi(t,X(t),Y(t),Z(t)) + \gamma\eta(t))\mathrm{d}t + \sigma\mathrm{d}W(t) \\ \mathrm{d}X(t) &= \phi(t,X(t),Y(t),Z(t))\mathrm{d}t \\ \mathrm{d}Y(t) &= \eta(t)\mathrm{d}t \end{aligned}$$

The Stochastic Optimal Control Problem

The HJB equation for player 2

$$\begin{aligned} V_t &- \frac{1}{4\lambda}(z + \lambda\phi(t,x,y,z) + V_y + \gamma V_z)^2 \\ &+ \phi(t,x,y,z)(V_x + \gamma V_z) + \frac{1}{2}\sigma^2 V_{zz} = 0 \end{aligned}$$

for $t \in [0,T]$, $x \in [0,x_0]$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, and where

$$\psi(t,x,y,z) = -\frac{1}{2\lambda}\left(z + \lambda\phi(t,x,y,z) + V_y + \gamma V_z\right)$$

Clean up the entanglement of $\phi(\cdots)$ and $\psi(\cdots)$, we get

$$\begin{aligned}
\phi(t,x,y,z) &= -\frac{1}{3\lambda}\left(z+2U_x-V_y+2\gamma U_z-\gamma V_z\right)_+ \\
&= -\frac{1}{3\lambda}\left(\delta(t,x,y,z)\right)_+ \\
\psi(t,x,y,z) &= \begin{cases} -\frac{1}{3\lambda}(z-U_x+2V_y-\gamma U_z+2\gamma V_z) \\ \qquad \text{if} \quad \delta(t,x,y,z)>0 \\ -\frac{1}{2\lambda}(z+V_y+\gamma V_z) \\ \qquad \text{if} \quad \delta(t,x,y,z)\le 0 \end{cases}
\end{aligned}$$

where $\delta(t,x,y,z) := z+2U_x-V_y+2\gamma U_z-\gamma V_z$.

The Nash-Equilibrium of This Game

$$\begin{aligned} \text{Define} \quad \delta(t,x,y,z) &:= \quad z + 2U_x - V_y + 2\gamma U_z - \gamma V_z \\ \text{and} \quad \delta^*(t,x,y,z) &:= \quad z + 2V_y - U_x + 2\gamma V_z - \gamma U_z \end{aligned}$$

where $\delta(t,x,y,z) > 0$,

$$\begin{cases} -U_t = & -\frac{1}{3\lambda}\delta^*(t,x,y,z)(U_y + \gamma U_z) \\ & -\frac{1}{9\lambda}\left(\delta(t,x,y,z)\right)^2 + \frac{1}{2}\sigma^2 U_{zz} \\ -V_t = & -\frac{1}{3\lambda}\delta(t,x,y,z)(V_x + \gamma V_z) \\ & -\frac{1}{9\lambda}\left(\delta^*(t,x,y,z)\right)^2 + \frac{1}{2}\sigma^2 V_{zz} \end{cases}$$

where $\delta(t,x,y,z) \le 0$,

$$\begin{cases} -U_t = & -\frac{1}{2\lambda}(z + V_y + \gamma V_z)(U_y + \gamma U_z) + \frac{1}{2}\sigma^2 U_{zz} \\ -V_t = & -\frac{1}{4\lambda}(z + V_y + \gamma V_z)^2 + \frac{1}{2}\sigma^2 V_{zz} \end{cases}$$

Numerical Analysis of the NE

In order to obtain stable numerical solutions, let us induce some viscosity condiments to the numerical scheme when solving the PDE system.
For the higher-order partials, instead of $\sigma^2 U_{zz}$
consider $\sigma^2 U_{zz} + \frac{1}{2}\epsilon\sigma^2 U_{xx} + \epsilon\sigma^2 U_{yy}$
and let $\epsilon \to 0$
Key verification: the numerical solution obtained is not sensitive at all to the choice of ϵ

Numerical Analysis of the NE

A trial example: the after-story of the liquidity trading game The HJB equation for the 2nd player during the sequel period $[T, \bar{T}]$

$$V_t + \min\left\{\lambda\eta^2 + (z + V_y + \gamma V_z)\eta \,|\eta \leq 0\right\}$$

$$+ \frac{1}{2}\epsilon\sigma^2 V_{yy} + \frac{1}{2}\sigma^2 V_{zz} = 0$$

$$-V_t = -\frac{1}{4\lambda}\left[(z + V_y + \gamma V_z)_+\right]^2 + \frac{1}{2}\sigma^2 V_{zz} + \frac{1}{2}\epsilon\sigma^2 V_{yy}$$

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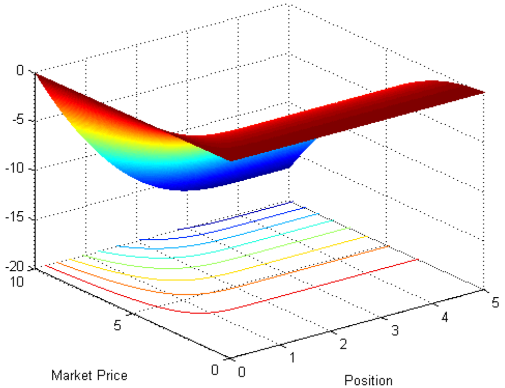


Figure: The numerical solution almost does not depend on the choice of coefficient for the artificial viscosity term. Here, $\epsilon = 2.5 * 10^{-3}$

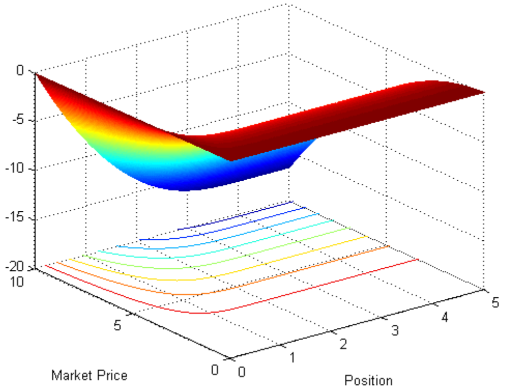
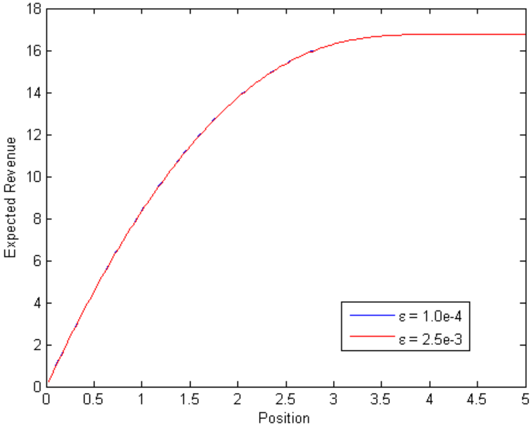


Figure: The numerical solution almost does not depend on the choice of coefficient for the artificial viscosity term. Here, $\epsilon = 1.0 * 10^{-4}$



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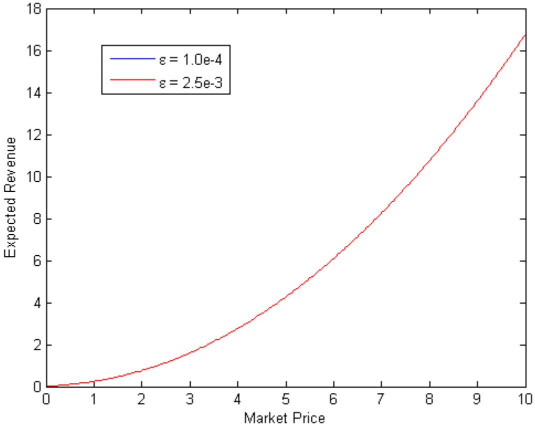

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Volatility Does Enter the Picture and Make A Difference

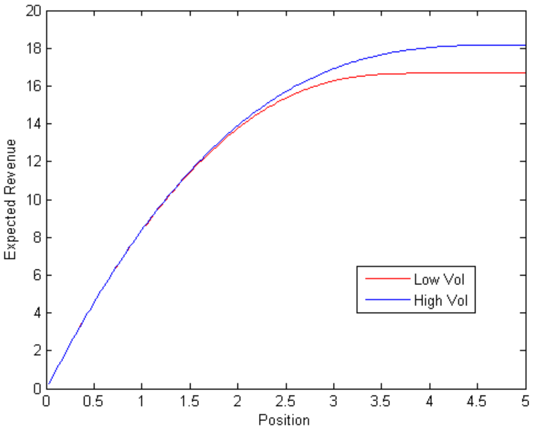

Figure: The terminal value function for the predator under different volatility levels

Volatility Does Enter the Picture and Make A Difference

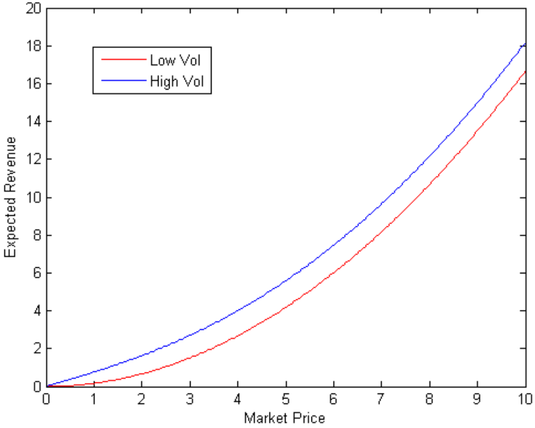

Figure: The terminal value function for the predator under different volatility levels

Summary and Ongoing Work

- Strategic interplay is an important source for Market Liquidity behaviors
- Adopt a reasonable market impact model, and think in *volume time*
- Closed-Loop strategies guarantee subgame perfectness, and usher volatility into the picture
- Numerical analysis of the NE of such a liquidity trading game

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The End

THANK YOU