Basis risk with random parameters

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Related literature

- Basis risk models:
  - Davis [3], Henderson [5], MM [8, 9, 10], Musiela and Zariphopoulou [11], Ankirchner, Imkeller and Reis [1]
- Exponential utility maximisation with random endowment:
  - Delbaen et al [4], Mania and Schweizer [7]
- Asymptotic analysis of utility-based prices for small numbers of claims:
  - Kramkov and Sîrbu [6]
- Partial information models:
  - Rogers [13], Björk, Davis and Landén [2]

Random parameter basis risk model

- $(\Omega, \mathcal{F}, P)$, $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$
- $(S, Y) = (S_t, Y_t)_{0 \le t \le T}$

$$dS_t = \sigma_t^S S_t(\lambda_t^S dt + dB_t^S), \quad dY_t = \sigma_t^Y Y_t(\lambda_t^Y dt + dB_t^Y)$$

  where

$$B^Y = \rho_t B^S + \sqrt{1 - \rho_t^2} Z^S, \quad \rho_t \in [-1, 1]$$

- $\mathbb{F}$-adapted parameters
- $r = 0$
- σ^S, σ^Y, ρ also $\mathbb{F}$-adapted
- Markovian model: $\nu_t \equiv \nu(t, S_t, Y_t)$, for $\nu \in \{\sigma^S, \sigma^Y, \lambda^S, \lambda^Y, \rho\}$
- Claim $C(Y_T) \ge 0$, bounded, $\mathcal{F}_T$-measurable

Exponential valuation and hedging

Fot $t \in [0, T]$, given $X_t = x$, portfolio wealth process is

$$X_u = x + \int_t^T \theta_u dS_u = x + \int_t^T \pi_u(\lambda_u^S du + dB_u^S), \quad t \le u \le T$$

where $\pi := \theta S$. Denote by Θ (or Π) the set of admissible θ (or π)

$$U(x) := \exp(-\alpha x), \quad x \in \mathbb{R}, \quad \alpha > 0$$

Given $(X_t, S_t, Y_t) = (x, s, y)$, primal value function is

$$u^C(t, x, s, y) := \sup_{\pi \in \Pi} E_{t,x,s,y}[U(X_T - C(Y_T))] \tag{1}$$

Indifference price $p(t, s, y)$ defined by

$$u^C(t, x + p(t, s, y), s, y) = u^0(t, x, s, y)$$

Denote optimal strategy for (1) by π^C. Optimal hedging strategy $\pi^{(H)}$ defined by

$$\pi^{(H)} := \pi^C - \pi^0$$

Entropy and admissibility

$$\mathbb{P}_e := \{Q \sim P | S \text{ is a local } (Q, \mathbb{F})\text{-martingale}\}$$

$$\mathbb{P}_{e,f} := \{Q \in \mathbb{P}_e | H(Q,P) < \infty\} \neq \emptyset$$

$$H(Q,P) := E\left[\frac{dQ}{dP} \log \frac{dQ}{dP}\right], \quad (\text{if finite, else } H(Q,P) := \infty)$$

For $Q \in \mathbb{P}_{e,f}$, define P-martingale Γ^Q by

$$\Gamma_t^Q := \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(-\lambda^S \cdot B^S - \psi \cdot Z^S), \quad 0 \le t \le T$$

Admissible strategies

$$\Theta := \{\theta | (\theta \cdot S) \text{ is a } (Q, \mathbb{F})\text{-martingale for all } Q \in \mathbb{P}_{e,f}\}$$

Duality

Dual value function

$$\tilde{u}(t,\eta,s,y) := \inf_{Q\in\mathbb{P}_{e,f}} E_{t,s,y}\left[\widetilde{U}\left(\eta\frac{\Gamma_T^Q}{\Gamma_t^Q}\right) - \eta\frac{\Gamma_T^Q}{\Gamma_t^Q}C(Y_T)\right]$$

where, for exponential utility

$$\widetilde{U} = \frac{\eta}{\alpha}\left[\log\left(\frac{\eta}{\alpha}\right) - 1\right]$$

Hence

$$\tilde{u}(t,\eta,s,y) = \widetilde{U}(\eta) + \frac{\eta}{\alpha}H^C(t,s,y)$$

where

$$H^C(t,s,y) := \inf_{Q\in\mathbb{P}_{e,f}} E^Q_{t,s,y}\left[\log\left(\frac{\Gamma_T^Q}{\Gamma_t^Q}\right) - \alpha C(Y_T)\right]$$

Primal and dual value functions conjugate, so primal problem has representation

$$u^C(t,x,s,y) = -\exp\left(-\alpha x - H^C(t,s,y)\right) \tag{2}$$

Dual control problem

$$B_t^{S,Q} := B_t^S + \int_0^t \lambda_u^S du, \quad Z_t^{S,Q} := Z_t^S + \int_0^t \psi_u du, \quad 0 \le t \le T$$

($\psi \equiv 0$ corresponds to Q^M). Then, for $Q \in \mathbb{P}_{e,f}$,

$$E_{t,s,y}^Q \log \frac{\Gamma_T^Q}{\Gamma_t^Q} = E_{t,s,y}^Q \frac{1}{2} \int_t^T \left[(\lambda_u^S)^2 + \psi_u^2 \right] du < \infty \qquad (3)$$

Let Ψ denote set of ψ such that (3) is satisfied. Then H^C is value function of stochastic control problem

$$H^C(t,s,y) := \inf_{\psi \in \Psi} E_{t,s,y}^Q \left[\frac{1}{2} \int_t^T \left[(\lambda_u^S)^2 + \psi_u^2 \right] du - \alpha C(Y_T) \right] \qquad (4)$$

where, under $Q \in \mathbb{P}_{e,f}$, state variables S, Y follow

$$\begin{aligned} dS_t &= \sigma_t^S S_t dB_t^{S,Q} \\ dY_t &= \sigma_t^Y Y_t \left[(\lambda_t^Y - \rho_t \lambda_t^S - \sqrt{1-\rho_t^2}\psi_t) dt + dB_t^{Y,Q} \right] \\ B_t^{Y,Q} &= \rho_t B_t^{S,Q} + \sqrt{1-\rho_t^2} Z_t^{S,Q} \end{aligned}$$

Dual representation of indifference price

- For $C \equiv 0$,

$$H^0(0,\cdot,\cdot) = H(Q^E, P)$$

- Write $H^0(t,s,y) \equiv H^E(t,s,y)$, the "minimal entropy" value function. Then

$$u^0(t,x,s,y) = -\exp\left(-\alpha x - H^E(t,s,y)\right) \tag{5}$$

Indifference price definition plus (2) and (5) give

$$u^C(t,x,s,y) = u^0(t,x,s,y)\exp\left(\alpha p(t,s,y)\right)$$

and indifference price has entropic representation

$$-\alpha p(t,s,y) = H^C(t,s,y) - H^E(t,s,y)$$

Optimal hedging strategy

Theorem

Optimal hedge: hold $\theta_t^{(H)}$ shares of S_t at $t \in [0, T]$, where

$$\theta_t^{(H)} = \left(\frac{\partial p}{\partial s}(t, S_t, Y_t) + \rho(t, S_t, Y_t) \frac{\sigma^Y(t, S_t, Y_t)}{\sigma^S(t, S_t, Y_t)} \frac{Y_t}{S_t} \frac{\partial p}{\partial y}(t, S_t, Y_t) \right)$$

Remark

Additional term $p_s(t, S_t, Y_t)$ compared with earlier studies. Partial information model of Section 3 is of this form, and $p_s(t, S_t, Y_t)$ reflects additional risk induced by parameter uncertainty.

Proof.

Use HJB equation associated with primal the value function

$$\frac{\partial u^C}{\partial t} + \max_{\pi} \mathcal{A}_{X,S,Y} u^C = 0$$

Compute optimal Markov control and use separable form (2) of value function, to obtain optimal strategy as $\pi_t^C = \pi^C(t, S_t, Y_t)$, where

$$\pi^C(t,s,y) = \frac{\lambda^S}{\sigma^S \alpha} - \frac{1}{\alpha}\left(s(H_s^E - \alpha p_s) + \rho \frac{\sigma^Y}{\sigma^S} y(H_y^E - \alpha p_y) \right)$$

with similar form for $C = 0$. Apply definition of optimal hedging strategy: $\theta^{(H)} S \equiv \pi^{(H)} = \pi^C - \pi^0$, to get result

□

(Smoothness of value function proven using methods in Pham [12])

Stochastic control problem for $p(t,s,y)$ when $Q^E = Q^M$

- Suppose $\lambda_t^S \equiv \lambda^S(t,S_t)$, and $\sigma_t^S \equiv \sigma^S(t,S_t)$
- Then infimum in (4) for $C = 0$ is achieved by $\psi = 0$, so

$$Q^E = Q^M$$

and $H^0(t,s,y) = H^E(t,s)$ is given by

$$H^E(t,s) = E_{t,s}^{Q^M}\left[\frac{1}{2}\int_t^T (\lambda_u^S)^2 du\right]$$

Indifference price $p(t,s,y)$ has stochastic control representation

$$p(t,s,y) = \sup_{\psi\in\Psi} E_{t,s,y}^{Q}\left[C(Y_T) - \frac{1}{2\alpha}\int_t^T \psi_u^2 du\right]$$

subject to state dynamics

$$\begin{aligned} dS_t &= \sigma^S(t,S_t)S_t dB_t^{S,Q} \\ dY_t &= \sigma_t^Y Y_t\left[(\lambda_t^Y - \rho_t\lambda_t^S - \sqrt{1-\rho_t^2}\psi_t)dt + dB_t^{Y,Q}\right] \end{aligned}$$

Indifference price PDE

- Define $\phi(t,s,y) = \sqrt{1-\rho^2}\sigma^Y y p_y$. HJB equation for $p(t,s,y)$ is

$$p_t + \mathcal{A}^{Q^M}_{S,Y} p + \max_{\psi}\left[-\frac{1}{2\alpha}\psi^2 - \phi\psi\right] = 0, \quad p(T,s,y) = C(y)$$

  Optimal Markov control is $\psi^C_t \equiv \psi^C(t, S_t, Y_t)$, where

$$\psi^C(t,s,y) = -\alpha\phi(t,s,y)$$

  Note that $\alpha \to 0 \Rightarrow \psi^C \to \psi^0 = 0$

- Hence p solves semi-linear PDE

$$p_t + \mathcal{A}^{Q^M}_{S,Y} p + \frac{1}{2}\alpha\phi^2 = 0, \quad p(T,s,y) = C(y)$$

  For $\alpha \to 0$, obtain linear PDE for marginal price p^M, and

$$p^M(t,s,y) := \lim_{\alpha\to 0} p(t,s,y) = E^{Q^M}_{t,s,y} C(Y_T)$$

Hedging error (residual risk) process

$$R_t := p(0, S_0, Y_0) + \int_0^t \theta_u^{(H)} dS_u - p(t, S_t, Y_t), \quad R_0 = 0, \quad 0 \le t \le T$$

Itô and PDE for $p(t, s, y)$ gives, with $\phi_t \equiv \phi(t, S_t, Y_t)$,

$$dR_t = \frac{1}{2}\alpha\phi_t^2 dt - \phi_t dZ_t^S$$

Define

$$A_t := -\exp(-\alpha R_t), \quad 0 \le t \le T$$

Then

$$A_t = -\mathcal{E}(\alpha\phi \cdot Z^S)_t, \quad 0 \le t \le T$$

so A is a Q^M-martingale (and also a P-martingale)

Payoff decomposition and price representation

Integrate SDE for R over $[t, T]$ and use definition of R (equivalently, directly use Itô and PDE for $p(t, s, y)$ over $[t, T]$):

$$C(Y_T) = p(t, S_t, Y_t) - \frac{1}{2}\alpha \int_t^T \phi_u^2 du + \int_t^T \phi_u dZ_u^S + \int_t^T \theta_u^{(H)} dS_u \qquad (6)$$

Expectation under Q^M given $(S_t, Y_t) = (s, y)$ gives

Lemma

Indifference price satisfies

$$p(t, s, y) = p^M(t, s, y) + \frac{1}{2}\alpha E_{t,s,y}^{Q^M} \int_t^T \phi^2(u, S_u, Y_u) du \qquad (7)$$

with

$$\phi(t, S_t, Y_t) = \sqrt{1 - \rho_t^2}\sigma_t^Y Y_t p_y(t, S_t, Y_t), \quad 0 \leq t \leq T$$

Föllmer-Schweizer-Sondermann decomposition

Let $\alpha \to 0$ in payoff decomposition (6) (or directly use PDE for p^M and Itô) to obtain

$$C(Y_T) = p^M(t, S_t, Y_t) + \int_t^T \phi_u^M dZ_u^S + \int_t^T \theta_u^M dS_u$$

where

$$\theta_t^M := p_s^M(t, S_t, Y_t) + \rho_t \frac{\sigma_t^Y}{\sigma_t^S} \frac{Y_t}{S_t} p_y^M(t, S_t, Y_t), \quad 0 \le t \le T$$

is the marginal hedging strategy and

$$\phi_t^M := \sqrt{1 - \rho_t^2} \sigma_t^Y Y_t p_y^M(t, S_t, Y_t), \quad 0 \le t \le T$$

Small risk aversion expansion

Denote

$$v(t,s,y) := \mathrm{var}^{Q^M}_{t,s,y}[C(Y_T)]$$

Theorem

The indifference pricing function $p(t,s,y)$ has the asymptotic expansion

$$p(t,s,y) = p^M(t,s,y) + \frac{1}{2}\alpha\left[v(t,s,y) - E^{Q^M}_{t,s,y}\langle(\theta^M \cdot S)\rangle_{[t,T]}\right] + O(\alpha^2)$$

Proof.

Write

$$p(t,s,y) = p^M(t,s,y) + \alpha p^{(1)}(t,s,y) + O(\alpha^2)$$

Apply to indifference price representation (7) to obtain

$$p^{(1)}(t,s,y) = \frac{1}{2} E^{Q^M}_{t,s,y} \int_t^T (\phi^M_u)^2 du$$

But from FSS decomposition, compute

$$v(t,s,y) = E^{Q^M}_{t,s,y} \int_t^T \left[(\phi^M_u)^2 + (\theta^M_u)^2 \right] du$$

and result follows

□

Higher order corrections computable

Invariance principle for Malliavin calculus

- Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$
- $\Omega = C([0,T])$, $\omega : [0,T] \to \mathbb{R}$, $W_t(\omega) = \omega(t)$ is BM and P is Wiener measure

For continuous, bounded square-integrable process φ, define

$$\Phi := \int_0^t \varphi_u^2 du, \quad 0 \leq t \leq T$$

For $\epsilon \in \mathbb{R}$, define measure P^ϵ on $(\Omega, \mathcal{F})$ by

$$\frac{dP^\epsilon}{dP} = \Gamma^\epsilon := \mathcal{E}(\epsilon\varphi \cdot W)_T$$

Under P^ϵ, $W - \epsilon\Phi$ is BM, so for square-integrable Brownian functional F ($\mathcal{F}_T$-measurable mapping $F : \Omega \to \mathbb{R}$ with $EF^2(W) < \infty$), we have invariance principle

$$EF(W) = E^\epsilon F(W - \epsilon\Phi) = E[F(W - \epsilon\Phi)\Gamma^\epsilon] \tag{8}$$

Malliavin calculus arises because we can differentiate (8) w.r.t. ϵ at $\epsilon = 0$

Application to indifference price optimisation problem

Simplify notation: take $t = 0$, relabel $Q \to P$

$$p = \sup_{\psi} E\left[C(Y_T) - \frac{1}{2\alpha}\int_0^T \psi_t^2 dt\right]$$

subject to

$$\begin{aligned} dY_t &= \sigma_t^Y Y_t\left[\left(a_t - \sqrt{1-\rho_t^2}\psi_t\right)dt + \rho_t dB_t + \sqrt{1-\rho_t^2}dZ_t\right] \\ dS_t &= \sigma_t^S S_t dB_t \end{aligned}$$

Idea is to consider ψ as a perturbation: write

$$\epsilon\varphi_t := -\psi_t$$

for some small parameter ϵ. Then

$$\frac{\psi^2}{\alpha} = \frac{\epsilon^2}{\alpha}\varphi^2 = \varphi^2, \quad \text{if we choose } \epsilon^2 := \alpha$$

Re-formulated problem

For simplicity, take $\sigma^S, \sigma^Y \rho$ constant. Then control problem is

$$p = \sup_{\varphi} E\left[C(Y_T^\epsilon) - \frac{1}{2}\int_0^T \varphi_t^2 dt\right]$$

subject to

$$\begin{aligned} dY_t^\epsilon &= \sigma^Y Y_t^\epsilon \left[a(t, S_t, Y_t^\epsilon)dt + \rho dB_t + \sqrt{1-\rho^2}(dZ_t + \epsilon\varphi_t dt)\right] \\ dS_t &= \sigma^S S_t dB_t \end{aligned}$$

where we write Y^ϵ to emphasise dependence on ϵ

Invariance principle

Look for measure P^ϵ such that

$$\mathrm{Law}\,(Y^\epsilon; P) = \mathrm{Law}\,\left(Y^0; P^\epsilon\right)$$

So define P^ϵ by

$$\frac{dP^\epsilon}{dP} = \mathcal{E}(\epsilon\phi \cdot Z)_T =: \Gamma^\epsilon$$

Then

$$\frac{1}{\epsilon}\left[EC(Y_T^\epsilon) - EC(Y_T^0)\right] = \frac{1}{\epsilon}E\left[(\Gamma^\epsilon - 1)\, C(Y_T^0)\right] \tag{9}$$

Lemma

Function $\epsilon \to EC(Y_T^\epsilon)$ *is differentiable at* $\epsilon = 0$*, with*

$$\frac{d}{d\epsilon} EC(Y_T^\epsilon)|_{\epsilon=0} = E\left[C(Y_T^0)\int_0^T \varphi_t dZ_t\right] \tag{10}$$

Proof.

Use

$$\frac{1}{\epsilon}(\Gamma^\epsilon - 1) \to \int_0^T \varphi_t dZ_t, \quad \text{in } L_2 \text{ as } \epsilon \to 0 \tag{11}$$

Establishes (10) in view of (9) and C bounded

□

Then, from (10)

$$E\left[C(Y_T^\epsilon) - \frac{1}{2}\int_0^T \varphi_t^2 dt\right] = EC(Y_T^0) + E\left[\epsilon C(Y_T^0)\int_0^T \varphi_t dZ_t - \frac{1}{2}\int_0^T \varphi_t^2 dt\right] + \ldots \tag{12}$$

Recall FSS decomposition, which in the current notation reads as

$$C(Y_T^0) = EC(Y_T^0) + (\phi^M \cdot Z)_T + (\theta^M \cdot S)_T$$

Then (12) becomes

$$E\left[C(Y_T^\epsilon) - \frac{1}{2}\int_0^T \varphi_t^2 dt\right] = EC(Y_T^0) + E\left[\int_0^T \left(\epsilon\phi_t^M \varphi_t - \frac{1}{2}\varphi_t^2\right) dt\right] + \ldots$$

Maximise by choosing $\varphi = \epsilon\phi^M$, to obtain

$$p = EC(Y_T^0) + \frac{1}{2}\epsilon^2 E\int_0^T (\phi_t^M)^2 dt$$

and result follows from FSS decomposition, since

$$\mathrm{var}[C(Y_T^0)] = E\int_0^T \left[(\phi_t^M)^2 + (\theta_t^M)^2\right] dt$$

Lognormal model under partial information

- $\sigma^S > 0, \sigma^Y > 0, \rho \in [-1, 1]$, known constants, inferred from $\langle S \rangle, \langle Y \rangle, \langle S, Y \rangle$
- λ^S, λ^Y are $\mathcal{F}_0$-measurable random variables, so would be known constants if agent had access to filtration $\mathbb{F}$
- In full information (completely observable) model, hedger uses $\mathbb{F}$-adapted strategies
- In partial information model, hedger uses $\widehat{\mathbb{F}}$-adapted strategies, where $\widehat{\mathbb{F}}$ is filtration generated by S, Y

Perfect correlation case

If correlation perfect $\rho = 1$, $\lambda^Y = \lambda^S$, and perfect hedge is

$$\Delta_t^{(\mathrm{BS})} = \frac{\sigma^Y}{\sigma^S}\frac{Y_t}{S_t}\frac{\partial}{\partial y}\mathrm{BS}(t, Y_t; \sigma^Y), \quad 0 \leq t \leq T$$

BS-style hedge does not require knowledge of λ^S, λ^Y

Completely observable incomplete case

If $\rho \neq 1$, indifference price at $t \in [0, T]$, is $p^F(t, Y_t)$, where

$$p^F(t,y) = \frac{1}{\alpha(1-\rho^2)} \log E^{Q^M}\left[\exp\left(\alpha(1-\rho^2)C(Y_T)\right)\middle|\, Y_t = y\right] \qquad (13)$$

Under Q^M, Y follows

$$dY_t = \sigma^Y Y_t[(\lambda^Y - \rho\lambda^S)dt + dB_t^{Y,Q^M}$$

Optimal hedging strategy Δ_t^F given by

$$\Delta_t^F = \rho\frac{\sigma^Y}{\sigma^S}\frac{Y_t}{S_t}\frac{\partial p^F}{\partial y}(t, Y_t), \quad 0 \leq t \leq T$$

Asymptotic expansion of (13) in powers of $\epsilon := \alpha(1-\rho^2)$

$$p^F(t,y) = E^{Q^M}[h(Y_T)|Y_t = y] + \frac{1}{2}\epsilon.\mathrm{var}^{Q^M}[h(Y_T)|Y_t = y] + O(\epsilon^2)$$

Requires knowledge of λ^S, λ^Y

Partial information case

λ^S, λ^Y are random variables with some prior distribution

$$\xi_t^S := \frac{1}{\sigma^S}\int_0^t \frac{dS_u}{S_u} = \lambda^S t + B_t^S, \quad \xi_t^Y := \frac{1}{\sigma^Y}\int_0^t \frac{dY_u}{Y_u} = \lambda^Y t + B_t^Y$$

or

$$\xi_t^S = \frac{1}{\sigma^S}\log\left(\frac{S_t}{S_0}\right) + \frac{1}{2}\sigma^S t, \quad \xi_t^Y = \frac{1}{\sigma^Y}\log\left(\frac{Y_t}{Y_0}\right) + \frac{1}{2}\sigma^Y t$$

Consider

$$\Xi_t := \begin{pmatrix}\xi_t^S \\ \xi_t^Y\end{pmatrix}, \quad 0 \le t \le T,$$

as observation process in Kalman-Bucy filter: noisy observations of "signal process"

$$\Lambda := \begin{pmatrix}\lambda^S \\ \lambda^Y\end{pmatrix}$$

Prior

Observation filtration $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \le t \le T}$

$$\widehat{\mathcal{F}}_t = \sigma(\xi^S_u, \xi^Y_u; 0 \le u \le t), \quad 0 \le t \le T$$

Assume Gaussian prior:

$$\mathrm{Law}(\Lambda | \widehat{\mathcal{F}}_0) = \mathrm{N}(\Lambda_0, V_0)$$

$$\Lambda_0 = \begin{pmatrix} \lambda^S_0 \\ \lambda^Y_0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} \mathrm{v}^S_0 & c_0 \\ c_0 & \mathrm{v}^Y_0 \end{pmatrix}, \quad c_0 := \rho \min(\mathrm{v}^S_0, \mathrm{v}^Y_0) \tag{14}$$

- Motivation: agent uses data before time zero to make point estimate of Λ, and uses distribution of estimator as prior
- With historical data for ξ^S (ξ^Y) over interval t_S (t_Y), then unbiased estimator of Λ is Gaussian according to (14) with $\lambda^i_0 = \lambda^i$, and $\mathrm{v}^i_0 = 1/t_i$, for $i = S, Y$

Observation and signal SDEs

$$d\Xi_t = \Lambda dt + D d\mathbf{B}_t, \quad d\Lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \quad \mathbf{B}_t = \begin{pmatrix} B_t^S \\ Z_t^S \end{pmatrix}$$

Optimal filter $\widehat{\Lambda}_t := E[\Lambda|\widehat{\mathcal{F}}_t], 0 \leq t \leq T$, is two conditional expectations

$$\widehat{\lambda}_t^i := E[\lambda^i|\widehat{\mathcal{F}}_t], \quad 0 \leq t \leq T, \quad i = S, Y$$

Conditional variances and covariance

$$\begin{aligned} \mathrm{v}_t^i &:= E\left[(\lambda^i - \widehat{\lambda}_t^i)^2 \middle| \widehat{\mathcal{F}}_t\right], \quad 0 \leq t \leq T, \quad i = S, Y \\ \mathrm{c}_t &:= E\left[(\lambda^S - \widehat{\lambda}_t^S)(\lambda^Y - \widehat{\lambda}_t^Y) \middle| \widehat{\mathcal{F}}_t\right], \quad 0 \leq t \leq T \end{aligned}$$

Covariance matrix

$$V_t := \begin{pmatrix} \mathrm{v}_t^S & \mathrm{c}_t \\ \mathrm{c}_t & \mathrm{v}_t^Y \end{pmatrix}, \quad 0 \leq t \leq T$$

Well known that $(V_t)_{0\leq t\leq T}$ is *deterministic*. Introduce notation

$$m_t := \min(\mathrm{v}_t^S, \mathrm{v}_t^Y), \quad M_t := \max(\mathrm{v}_t^S, \mathrm{v}_t^Y), \quad b_t := \frac{M_t - \rho^2 m_t}{1-\rho^2}$$

Kalman-Bucy filter to converts the partial information model to an equivalent full information model

Proposition

Effective full information model on $(\Omega, \widehat{\mathcal{F}}_T, \widehat{\mathbb{F}}, P)$

$$dS_t = \sigma^S S_t(\widehat{\lambda}^S_t dt + d\widehat{B}^S_t), \quad dY_t = \sigma^Y Y_t(\widehat{\lambda}^Y_t dt + d\widehat{B}^Y_t) \tag{15}$$

1. *For* $i, j \in \{S, Y\}$, *if* $m_0 = \mathrm{v}^i_0 < \mathrm{v}^j_0 = M_0$, *then*

$$\widehat{\lambda}^i_t = \frac{\lambda^i_0 + m_0\xi^i_t}{1 + m_0 t}, \quad \widehat{\lambda}^j_t - \rho\widehat{\lambda}^i_t = \frac{\widehat{\lambda}^j_0 - \rho\widehat{\lambda}^i_0 + b_0(\xi^j_t - \rho\xi^i_t)}{1 + b_0 t} \tag{16}$$

2. *If* $m_0 = \mathrm{v}^S_0 = \mathrm{v}^Y_0 = M_0$, *then*

$$\widehat{\lambda}^i_t = \frac{\lambda^i_0 + m_0\xi^i_t}{1 + m_0 t}, \quad i = S, Y \tag{17}$$

Proposition, continued..

Proposition

Functions $\mathrm{v}^S, \mathrm{v}^Y, c$ *given by*

$$\mathrm{v}^i_t = m_t, \quad \mathrm{v}^j_t = M_t, \quad c_t = \rho m_t, \quad \text{if } m_0 = \mathrm{v}^i_0 < \mathrm{v}^j_0 = M_0, \quad i,j \in \{S, Y\} \tag{18}$$

and

$$\mathrm{v}^S_t = \mathrm{v}^Y_t = m_t = M_t, \quad c_t = \rho m_t, \quad \text{if } m_0 = \mathrm{v}^S_0 = \mathrm{v}^Y_0 = M_0 \tag{19}$$

with

$$m_t = \frac{m_0}{1 + m_0 t}, \quad b_t = \frac{b_0}{1 + b_0 t}, \quad 0 \le t \le T \tag{20}$$

Proof

By Kalman-Bucy filter, $\widehat{\Lambda}$ satisfies

$$d\widehat{\Lambda}_t = V_t \left(DD^T\right)^{-1} (d\Xi_t - \widehat{\Lambda}_t dt) =: V_t \left(DD^T\right)^{-1} dN_t, \quad \widehat{\Lambda}_0 = \Lambda_0 \tag{21}$$

Innovations process N

$$N_t := \Xi_t - \int_0^t \widehat{\Lambda}_u du, \quad 0 \leq t \leq T$$

is $\widehat{\mathbb{F}}$-Brownian motion:

$$N_t = \begin{pmatrix} \widehat{B}_t^S \\ \widehat{B}_t^Y \end{pmatrix}, \quad \langle \widehat{B}^S, \widehat{B}^Y \rangle_t = \rho t, \quad 0 \leq t \leq T \tag{22}$$

Using this and

$$d\begin{pmatrix} S_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \sigma^S S_t \\ \sigma^Y Y_t \end{pmatrix} d\Xi_t$$

gives dynamics (15) of S, Y in observation filtration

Proof, continued

Matrix V_t satisfies Riccati equation

$$\frac{dV_t}{dt} = -V_t \left(DD^T\right)^{-1} V_t,$$

with V_0 given in (14). Then $F_t := V_t^{-1}$ satisfies Lyapunov equation

$$\frac{dF_t}{dt} = \left(DD^T\right)^{-1}$$

solved to give (18), (19) and (20). Use these in filtering equation (21) to get

1. For $i, j \in \{S, Y\}$, if $m_0 = \mathrm{v}_0^i < \mathrm{v}_0^j = M_0$,

$$\begin{aligned} d\widehat{\lambda}_t^i &= m_t d\widehat{B}_t^i = m_t(d\xi_t^i - \widehat{\lambda}_t^i dt), \quad \widehat{\lambda}_0^i = \lambda_0^i, \\ d(\widehat{\lambda}_t^j - \rho\widehat{\lambda}_t^i) &= b_t(d\widehat{B}_t^j - \rho d\widehat{B}_t^i) = b_t[d(\xi_t^j - \rho\xi_t^i) - (\widehat{\lambda}_t^j - \rho\widehat{\lambda}_t^i)dt] \end{aligned}$$

   with $\widehat{\lambda}_0^j = \lambda_0^j$

2. If $m_0 = \mathrm{v}_0^S = \mathrm{v}_0^Y = M_0$,

$$d\widehat{\lambda}_t^i = m_t d\widehat{B}_t^i = m_t(d\xi_t^i - \widehat{\lambda}_t^i dt), \quad \widehat{\lambda}_0^i = \lambda_0^i, \quad i = S, Y$$

From these SDEs obtain (16) and (17)

□

Summary

Abusing notation, we have

$$\begin{array}{lll} \widehat{\lambda}^S_t \equiv \widehat{\lambda}^S(t, S_t), & \widehat{\lambda}^Y_t \equiv \widehat{\lambda}^Y(t, S_t, Y_t), & \text{if } \mathrm{v}^S_0 < \mathrm{v}^Y_0 \\ \widehat{\lambda}^S_t \equiv \widehat{\lambda}^S(t, S_t), & \widehat{\lambda}^Y_t \equiv \widehat{\lambda}^Y(t, Y_t), & \text{if } \mathrm{v}^S_0 = \mathrm{v}^Y_0 \\ \widehat{\lambda}^S_t \equiv \widehat{\lambda}^S(t, S_t, Y_t), & \widehat{\lambda}^Y_t \equiv \widehat{\lambda}^Y(t, Y_t), & \text{if } \mathrm{v}^S_0 > \mathrm{v}^Y_0 \end{array} \tag{23}$$

$$\begin{array}{lll} d\widehat{\lambda}^S_t = m_t d\widehat{B}^S_t, & d\widehat{\lambda}^Y_t - \rho d\widehat{\lambda}^S_t = b_t(d\widehat{B}^Y_t - \rho d\widehat{B}^S_t), & \text{if } \mathrm{v}^S_0 < \mathrm{v}^Y_0 \\ d\widehat{\lambda}^S_t = m_t d\widehat{B}^S_t, & d\widehat{\lambda}^Y_t = m_t d\widehat{B}^Y_t, & \text{if } \mathrm{v}^S_0 = \mathrm{v}^Y_0 \\ d\widehat{\lambda}^Y_t = m_t d\widehat{B}^Y_t, & d\widehat{\lambda}^S_t - \rho d\widehat{\lambda}^Y_t = b_t(d\widehat{B}^S_t - \rho d\widehat{B}^Y_t), & \text{if } \mathrm{v}^S_0 > \mathrm{v}^Y_0 \end{array} \tag{24}$$

Intuition: estimation of drift of (geometric) Brownian motion depends only on length of time interval for which it is observed

Exponential hedging in effective full information model

- We have model with random drifts given by (23) and (24) (in limit $\mathrm{v}^S \to 0, \mathrm{v}^Y \to 0$, we recover standard full information model)
- So simply make replacements

$$\mathbb{F} \to \widehat{\mathbb{F}}, \quad B^i \to \widehat{B}^i, \quad \lambda^i \to \widehat{\lambda}^i, \quad i \in \{S, Y\}$$

  in Markovian basis risk model with random parameters
- For explicit results, consider case in which

$$\mathrm{v}_0^S \leq \mathrm{v}_0^Y \Leftrightarrow \widehat{\lambda}_t^S \equiv \widehat{\lambda}^S(t, S_t)$$

  so that $Q^E = Q^M$, and take σ^S, σ^Y, ρ constant
- Marginal price and hedge computable in closed form since, under Q^M, $\log Y_T$ is Gaussian

Lognormal distribution for Y_T

Proposition

Under Q^M*, conditional on* $S_t = s, Y_t = y$, $\log Y_T \sim \mathrm{N}(\mu, \Sigma^2)$*, where* $\mu \equiv \mu(t,s,y)$ *and* $\Sigma^2 \equiv \Sigma^2(t)$ *are given by*

$$
\begin{aligned}
mu(t,s,y) &= \log y + \sigma^Y \left(\widehat{\lambda}^Y(t,s,y) - \rho \widehat{\lambda}^S(t,s) - \frac{1}{2}\sigma^Y \right)(T-t) \\
\Sigma^2(t) &= \left[1 + (1-\rho^2) b_t (T-t) \right] (\sigma^Y)^2 (T-t)
\end{aligned}
$$

with $b_t = m_t$ *if* $\mathrm{v}_0^S = \mathrm{v}_0^Y$

Gives BS-style formulae for marginal price and hedge. Higher order corrections easily computable too

Proof.

Use Itô and SDEs for Y and $\widehat{\lambda}_t^Y - \rho\widehat{\lambda}_t^S$ under Q^M: for $t < T$,

$$\log \frac{Y_T}{Y_t} = \sigma^Y \int_t^T \left(\widehat{\lambda}_u^Y - \rho\widehat{\lambda}_u^S\right) du - \frac{1}{2}(\sigma^Y)^2(T-t) + \sigma^Y \int_t^T d\widehat{B}_u^{Y,Q^M} \qquad (25)$$

where $\widehat{B}^{Y,Q^M} = \rho\widehat{B}^{S,Q} + \sqrt{1-\rho^2}\widehat{Z}^S$. Dynamics of $\widehat{\lambda}_t^Y - \rho\widehat{\lambda}_t^S$ under Q^M are

$$d(\widehat{\lambda}_t^Y - \rho\widehat{\lambda}_t^S) = \sqrt{1-\rho^2} b_t d\widehat{Z}_t^S$$

Hence, for $u > t$, after changing the order of integration in a double integral, we have

$$\int_t^T \left(\widehat{\lambda}_u^Y - \rho\widehat{\lambda}_u^S\right) du = \left(\widehat{\lambda}_t^Y - \rho\widehat{\lambda}_t^S\right)(T-t) + \sqrt{1-\rho^2}\int_t^T b_u(T-u) d\widehat{Z}_u^S$$

Insert into (25) to yield result

□

Numerical experiments

- Simulation study: generate asset price paths over $[-t_0, T]$ (so take $\mathrm{v}_0^Y = \mathrm{v}_0^S$)
- Use data over $[-t_0, 0]$ to estimate drifts, and so set prior at time 0
- Sell put at time 0 for $p^M(0, S_0, Y_0)$ and optimally hedge over $[0, T]$, incorporating updating from filtering
- Generate terminal hedging error, and repeat over many paths to generate terminal hedging error distribution
- Compare with BS-style hedge, and also with results in absence of filtering

Parameters

$$
\begin{array}{lllll}
t_0 = 1, & \delta t = 1/504 & T = 1 & & \\
S_{-t_0} = 80, & Y_{-t_0} = 80 & & & \\
\lambda^S = 0.3, & \sigma^S = 0.2, & \lambda^Y = 0.4, & \sigma^Y = 0.25, & \rho = 0.75 \\
\alpha = 0.01, & K = 100 & & &
\end{array}
$$

Using marginal price and associated hedge

Table: Hedging error statistics (as percentage of premium): $\langle S_0 \rangle = 84.88$, $\langle Y_0 \rangle = 86.25$, $\langle p_0^M \rangle = 19.98$, $\langle \theta_0^M \rangle = -0.5885$; $\langle p_0^{\mathrm{BS}} \rangle = 19.96$, $\langle \Delta_0^{\mathrm{BS}} \rangle = -0.8397$; $\langle p_0^{\mathrm{NF}} \rangle = 19.75$, $\langle \Delta_0^{\mathrm{NF}} \rangle = -0.6284$

	Mean	SD	Median
Optimal Hedge	0.1948	0.5141	0.1834
BS Hedge	0.1143	0.6674	0.0873
Unfiltered Hedge	0.1613	0.5623	0.1567

With higher correlation, $\rho = 0.9$

Table: Hedging error statistics, $\rho = 0.9$. $\langle S_0 \rangle = 84.90$, $\langle Y_0 \rangle = 86.31$, $\langle p_0^M \rangle = 19.75$, $\langle \theta_0^M \rangle = -0.7325$; $\langle p_0^{\mathrm{BS}} \rangle = 19.91$, $\langle \Delta_0^{\mathrm{BS}} \rangle = -0.8414$; $\langle p_0^{\mathrm{NF}} \rangle = 19.64$, $\langle \Delta_0^{\mathrm{NF}} \rangle = -0.7547$

	Mean	SD	Median
Optimal Hedge	0.1416	0.3948	0.1014
BS Hedge	0.1116	0.4413	0.0678
Unfiltered Hedge	0.1226	0.4004	0.0846

Further questions

- General case, with random parameters and $Q^E \neq Q^M$
- Relationship with risk tolerance, and general representation for asymptotic expansions when $\mathrm{dom}(U) = \mathbb{R}$

References

S. Ankirchner, P. Imkeller, and G. Reis, *Pricing and hedging of derivatives based on non-tradable underlyings*.
Preprint, 2007.

T. Björk, M. H. A. Davis, and C. Landén, *Optimal investment under partial information*.
Preprint.

M. H. A. Davis, *Optimal hedging with basis risk*, in From stochastic calculus to mathematical finance, Y. Kabanov, R. Liptser, and J. Stoyanov, eds., Springer, Berlin, 2006, pp. 169–187.

F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker, *Exponential hedging and entropic penalties*, Math. Finance, 12 (2002), pp. 99–123.

V. Henderson, *Valuation of claims on nontraded assets using utility maximization*, Math. Finance, 12 (2002), pp. 351–373.

D. Kramkov and M. Sîrbu, *Asymptotic analysis of utility-based hedging strategies for small number of contingent claims*, Stochastic Process. Appl., 117 (2007), pp. 1606–1620.

M. Mania and M. Schweizer, *Dynamic exponential utility indifference valuation*, Ann. Appl. Probab., 15 (2005), pp. 2113–2143.

M. Monoyios, *Performance of utility-based strategies for hedging basis risk*, Quant. Finance, 4 (2004), pp. 245–255.

———, *Optimal hedging and parameter uncertainty*, IMA J. Manag. Math., 18 (2007), pp. 331–351.

M. Monoyios, *Marginal utility-based hedging of claims on non-traded assets with partial information*.
Submitted, 2008.

M. Musiela and T. Zariphopoulou, *An example of indifference prices under exponential preferences*, Finance Stoch., 8 (2004), pp. 229–239.

H. Pham, *Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints*, Appl. Math. Optim., 46 (2002), pp. 55–78.

L. C. G. Rogers, *The relaxed investor and parameter uncertainty*, Finance Stoch., 5 (2001), pp. 131–154.