Loss Control in Behavioural Portfolio Selection

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Behavioural Preference

- Reference point B
- S-shaped utility $u(\cdot)$: $u(x) = u_+(x^+) - u_-(x^-)$ with $u_\pm(\cdot)$ concave, $\uparrow$ and $u_\pm(0) = 0$
- Probability distortion $T_\pm(\cdot)$: $T_\pm(0) = 0, T_\pm(1) = 1$, $\uparrow$

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- Behavioural criterion: for a r.v. Y,

$$V(Y) = \int_0^{+\infty} u(y) d[-T_+(P(Y \geq y))] + \int_{-\infty}^0 u(y) d[T_-(P(Y \leq y))]$$

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- Behavioural Preference: compare $V(X - B)$

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  - Black-scholes market: $\rho_t = e^{-(r+\frac{(\mu-r)^2}{2\sigma^2})t-\frac{\mu-r}{\sigma}W(t)}$.

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  - Under some condition, the completeness can be relaxed.

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  - A contingent claim ξ at time T is obtainable from initial wealth x_0 if and only if
    - lower bounded, $E[\xi\rho_T] \leq x_0$.
- Denote $\rho = \rho_T$. Portfolio selection with criterion $\mathcal{U}(\cdot)$:

$$\begin{aligned} &\text{Maximize} \quad \mathcal{U}(X) \\ &s.t. \quad \begin{cases} X \text{ is lower bounded, } X \in \mathcal{A} \\ E[X\rho] \leq x_0 \end{cases} \end{aligned}$$

where $\mathcal{A}$ is a constraint on portfolio

Outline of this talk

- Unconstraint behavioural portfolio selection
- Behavioural portfolio selection with controlled loss
- Examples for the latter

Behavioural portfolio selection problem

$$\text{Maximize} \quad V(X - B)$$

$$s.t. \quad \begin{cases} X \text{ is lower bounded}, X \in \mathcal{A} \\ E[X\rho] \leq x_0 \end{cases}$$

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$$s.t. \quad \begin{cases} X \text{ is lower bounded}, X \in \mathcal{A} \\ E[X\rho] \leq x_0 \end{cases}$$

Suppose the reference is lower bounded. Rewrite the problem by changing variable $\tilde{X} = X - B$,

$$\text{Maximize} \quad V_+(\tilde{X}^+) - V_-(\tilde{X}^-)$$

$$s.t. \quad \begin{cases} \tilde{X} \text{ is lower bounded}, X \in \mathcal{A} \\ E[\tilde{X}\rho] \leq \tilde{x}_0 := x_0 - E[\rho B] \end{cases}$$

where $V_\pm(Y) = \int_0^{+\infty} T_\pm(P(u_\pm(y) \geq y))dy$.

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- $\{\tilde{X}^* \geq 0\} = \{\rho \leq c^*\}$, $\tilde{x}^*_+ := E[(\tilde{X}^*)^+\rho] \geq \tilde{x}_0^+$.

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$(\tilde{X}^*)^+$ is optimal for

$$\max \quad V_+(\tilde{X}_+)$$

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$(\tilde{X}^*)^-$ is optimal for

$$\min \quad V_-(\tilde{X}_-)$$
$$s.t. \begin{cases} \tilde{X}_- \geq 0, \ \tilde{X}_- \text{ is bounded} \\ \tilde{X}_- = 0 \text{ when } \rho < c^* \\ E[\tilde{X}_-\rho] = \tilde{x}_+^* - \tilde{x}_0 \end{cases}$$

Unconstraint case — Splitting

- Splitting: for any $c \in (\operatorname{essinf}\rho, \operatorname{esssup}\rho)$, $\tilde{x}_+ \geq \tilde{x}_0^+$, solve the following problems to get their value function $v_\pm(c, \tilde{x}_+)$

$$\max \quad V_+(\tilde{X}_+)$$
$$s.t. \begin{cases} \tilde{X}_+ \geq 0 \\ \tilde{X} = 0 \text{ when } \rho > c \\ E[\tilde{X}_+ \rho] = \tilde{x}_+ \end{cases}$$

(Positive Part Problem)

$$\min \quad V_-(\tilde{X}_-)$$
$$s.t. \begin{cases} \tilde{X}_- \geq 0,\ \tilde{X}_- \text{ is bounded} \\ \tilde{X}_- = 0 \text{ when } \rho < c \\ E[\tilde{X}_- \rho] = \tilde{x}_+ - \tilde{x}_0 \end{cases}$$

(Negative Part Problem)

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$$\max \quad V_+(\tilde{X}_+) \quad s.t. \begin{cases} \tilde{X}_+ \geq 0 \\ \tilde{X} = 0 \text{ when } \rho > c \\ E[\tilde{X}_+ \rho] = \tilde{x}_+ \end{cases}$$

(*Positive Part Problem*)

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(*Negative Part Problem*)

- Then find the optimal splitting c^* and $\tilde{x}_+^*$ by solving
$\text{Maximize}_{c \in (\text{essinf}\,\rho, \text{esssup}\,\rho), \tilde{x}_+ \geq x_0^+} v_+(c, \tilde{x}_+) - v_-(c, \tilde{x}_+)$.

Recovery of optimal contingent claim

- If
  - $c^*, \tilde{x}_+^*$ is an optimal splitting
  - $\tilde{X}_+^*, \tilde{X}_-^*$ are optimal for the two subproblems respectively with parameters $c^*, \tilde{x}_+^*$,

  then $X = \tilde{X}_+^* \mathbf{1}_{\rho \leq c^*} - \tilde{X}_-^* \mathbf{1}_{\rho > c^*} + B$ is optimal

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  then $X = \tilde{X}_+^* \mathbf{1}_{\rho \leq c^*} - \tilde{X}_-^* \mathbf{1}_{\rho > c^*} + B$ is optimal
- If any of them fails to exist, then there is no optimal contingent claim

Positive part problem

Consider the problem

max-Y

$$\max \quad V_+(Y) = \int_0^{+\infty} T_+(P(u_+(Y) \geq y))dy$$

$$s.t. \quad \begin{cases} Y \geq 0 \\ E[Y\rho] = a \end{cases}$$

with constant $a > 0$.

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with constant $a > 0$.

Theorem 1 Suppose this problem admits an optimal solution Y^*, with cumulative distribution function $G(\cdot)$, then

$$Y^* = G^{-1}(1 - F_\rho(\rho)), \quad a.s.,$$

where $F_\rho(\cdot)$ is the distribution function of ρ.

Positive part problem

Denote $Z = 1 - F_\rho(\rho)$, then $Z \sim U(0,1)$, and the problem can be written into

$$\begin{aligned} \max \quad & v_1(G(\cdot)) = \int_0^{+\infty} T(P\{u_+(G^{-1}(Z)) > y\})dy \\ s.t. \quad & G(\cdot) \text{ is a CDF with } G(0) = 0 \\ & E[G^{-1}(Z)F_\rho^{-1}(1-Z)] = a \end{aligned}$$

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Change optimization variable from $G(\cdot)$ to $g(\cdot) = G^{-1}(\cdot)$,

$$\begin{aligned} \max \quad & \bar{v}_1(g(\cdot)) := E[u_+(g(Z))T'(1-Z)] \\ \text{(max-g)} \quad s.t. \quad & E[g(Z)F_\rho^{-1}(1-Z)] = a \\ & g(\cdot) \text{ is a qunatile function with } g(0+) \geq 0 \end{aligned}$$

Positive part problem — The optimal solution

Assumption 1 : (1) $F_\rho(\cdot)$ is continuous; (2) $\frac{F_\rho^{-1}(\cdot)}{T_+'(\cdot)}$ is increasing on $[0,1]$; (3) $\liminf_{x\to+\infty}\frac{-xu_+''(x)}{u_+'(x)}>0$; (4) $E[u_+((u_+')^{-1}(\frac{\rho}{T'(F_\rho(\rho))}))T'(F_\rho(\rho))]<+\infty$.

Positive part problem — The optimal solution

Assumption 1 : (1) $F_\rho(\cdot)$ is continuous; (2) $\frac{F_\rho^{-1}(\cdot)}{T'_+(\cdot)}$ is increasing on $[0,1]$; (3) $\liminf_{x\to+\infty} \frac{-xu''_+(x)}{u'_+(x)} > 0$; (4) $E[u_+((u'_+)^{-1}(\frac{\rho}{T'(F_\rho(\rho))}))T'(F_\rho(\rho))] < +\infty$.

Theorem 2 Suppose Assumption 1 holds. Then for any $c \in (\text{essinf}\rho, \text{esssup}\rho]$ and $\tilde{x}_+ \geq \tilde{x}_0^+$, the optimal solution for the positive part problem is

$$\tilde{X}_+^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))})\mathbf{1}_{\rho \leq c}.$$

The optimal value is

$$v_+(c, \tilde{x}_+) = E[u_+((u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))}))T'_+(F(\rho))\mathbf{1}_{\rho \leq c}],$$

where λ is the unique one making $\tilde{X}_+^*$ feasible.

Negative part problem — Unconstraint case

Consider the problem

$$(\text{min-Y}) \quad \begin{array}{ll} \min & V_-(Y) = \int_0^{+\infty} T_-(P\{u_-(Y) > y\})dy \\ s.t. & E[Y\rho] = a, \;\; Y \geq 0 \end{array}$$

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Similar to the positive part problem, denoting $Z = F_\rho(\xi)$, we can translate the problem into

$$\text{(min-g)} \quad \begin{aligned} &\min \quad \bar{v}_2(g(\cdot)) := E[u_-(g(Z))T_-'(1-Z)] \\ &s.t. \quad g(\cdot) \text{ is a quantile function with } g(0+) \geq 0 \\ &\qquad\quad E[g(Z)F_\rho^{-1}(Z)] = a. \end{aligned}$$

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- If $g^*(\cdot)$ is optimal, then $Y^* := g^*(F_\rho(\rho))$ is optimal

Negative part problem — Unconstraint case

- Problem (min-g) is to *minimize* a *concave objective* over a convex set

Theorem 3 If Problem (min-g) admits optimal solutions, then one of them is in the form $g(x;c) = \frac{a}{E[\rho \mathbf{1}_{\rho \geq c}]} \mathbf{1}_{x \geq F_\rho(c)}$.

- $g^*(\cdot)$ is a two-piece step function
- Only need to solve the problem

$$\text{(minV-c)} \qquad \begin{array}{ll} \min & u_-\left(\frac{a}{E[\rho \mathbf{1}_{\rho \geq c}]}\right) T_-(P(\rho > c)) \\ s.t. & c \in [\operatorname{essinf} \rho, \operatorname{esssup} \rho) \end{array}$$

- The optimal solution Y^*, if exists, is *automatically bounded*.

Negative part problem — Unconstraint case

Theorem 4 For any $c \in [\mathrm{essinf}\rho, \mathrm{esssup}\rho)$, $\tilde{x}_+ > \tilde{x}_0^+$, the optimal value of the negative part problem is

$$v_-(c, \tilde{x}_+) = \inf_{c_2 \in [c, \mathrm{esssup}\rho)} u_-(\frac{\tilde{x}_+ - \tilde{x}_0}{E\rho\mathbf{1}_{\rho > c_2}})T_-(1 - F(c_2)).$$

Furthermore, if c_2^* achieves the optimal value $v_-(c, \tilde{x}_+)$, then

$$\tilde{X}_-^* = \frac{\tilde{x}_+ - \tilde{x}_0}{E\rho\mathbf{1}_{\rho > c_2^*}}\mathbf{1}_{\rho \geq c_2^*}.$$

Unconstraint case — Final optimal solution

The optimal splitting $c^*, \tilde{x}_+^*$ can be determined by

$$(\text{sp}) \quad \begin{array}{ll} \max & v_+(c, \tilde{x}_+) - u_-\left(\frac{\tilde{x}_+ - \tilde{x}_0}{E[\rho \mathbf{1}_{\rho > c}]}\right) T_-(1 - F(c)) \\ s.t. & \tilde{x}_+ \geq \tilde{x}_0, c \in [\text{essinf}\, \rho, \text{esssup}\, \rho) \end{array}$$

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Theorem 5 Suppose Assumption 1 hold.

(i) If $(c^*, \tilde{x}_+^*)$ is optimal for Problem (sp), then

$$X^* = (u'_+)^{-1}\left(\lambda \frac{\rho}{T'_+(F(\rho))}\right) \mathbf{1}_{\rho \leq c^*} - \frac{\tilde{x}_+^* - \tilde{x}_0}{E[\rho \mathbf{1}_{\rho > c^*}]} \mathbf{1}_{\rho > c^*} + B$$

is an optimal contingent claim.

Unconstraint case — Final optimal solution

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is an optimal contingent claim.

(ii) If there is no optimal $(c, \tilde{x}_+)$, then there is no optimal contingent claim.

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- An exogenous constraint $\tilde{X}^- \leq L$ controls the loss. (L is a positive number)
- $\tilde{X}^- \leq L$ is the same as $X \geq B - L$
  - No bankruptcy constraint if $L = B$
- Consider the problem with controlled loss

$$
\begin{array}{ll}
\text{Maximize} & V(\tilde{X}) \\
s.t. & \begin{cases} \tilde{X} \geq -L \\ E[\tilde{X}\rho] \leq \tilde{x}_0 \end{cases}
\end{array}
$$

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Negative part problem

- With the controlled loss
  - splitting of the problem still holds
  - positive part problem is formally the same
  - Difficulty exists in the negative part problem

$$
\text{(cnp)} \qquad \begin{array}{ll} \min & V_-(\tilde{X}_-) \\ s.t. & \begin{cases} \tilde{X}_- \geq 0, \ \tilde{X}_- \leq L \\ \tilde{X}_- = 0 \text{ when } \rho < c \\ E[\tilde{X}_- \rho] = \tilde{x}_+ - \tilde{x}_0 \end{cases} \end{array}
$$

where $(c, \tilde{x}_+)$ is a splitting.

Negative part problem with constraint

Consider the problem

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- By the same transformation, it is equivalent to solve

$$\min \quad \bar{v}_2(g(\cdot)) := E[u_-(g(Z))T'_-(1-Z)]$$
$$s.t. \quad \begin{cases} g(\cdot) \text{ is a quantile function with } g(0+) \geq 0 \\ g(\cdot) \leq L \text{ on } [0,1) \\ E[g(Z)F_\rho^{-1}(Z)] = a. \end{cases}$$

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- With the constraint $g(\cdot) \leq L$, the boundary of the feasible region changed.

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Theorem 6 If there are optimal $g(\cdot)$, then one of them is in the form $g(x; c1, c_2) = q(c_1, c_2; a)\mathbf{1}_{x \in [F_\rho(c_1), F_\rho(c_2))} + L\mathbf{1}_{x \geq F_\rho(c_2)}$, where $q(c_1, c_2; a) = \frac{a - LE[\rho \mathbf{1}_{\rho \geq c_2}]}{E[\rho \mathbf{1}_{\rho \in [c_1, c_2)}]}$.

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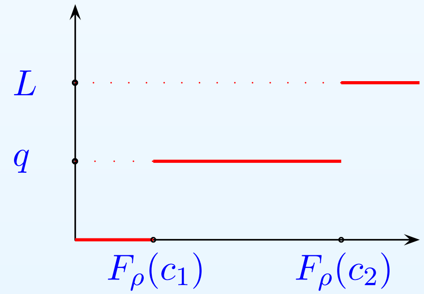

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- Only need to solve the problem

$$\text{(min-c)} \qquad \begin{array}{ll} \min & \bar{v}_2(g(\cdot; c_1, c_2)) \\ s.t. & \text{essinf}\rho \leq c_1 < c_2 \leq \text{esssup}\rho \end{array}$$

Negative part problem with constraint

Theorem 7 For any $c \in [\text{essinf}\rho, \text{esssup}\rho)$, $\tilde{x}_+ > \tilde{x}_0^+$, the optimal value of the Problem (cnp) is

$$v_-(c, \tilde{x}_+) = \inf_{c \le c_1 < c_2 \le \text{esssup}\rho)} v_3(c_1, c_2; c, \tilde{x}_+),$$

where

$$\begin{aligned} v_3(\cdots) &= u_-(q(c_1, c_2, \tilde{x}_+ - \tilde{x}_0))(T_-(P(\rho \ge c_2)) - T_-(P(\rho \ge c_1))) \\ &\quad + u_-(L)T_-(P(\rho \ge c_2)). \end{aligned}$$

Furthermore, if $v_-(c, x_+)$ is obtained at (c_1^*, c_2^*), then

$$\tilde{X}_-^* = q(c_1^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)\mathbf{1}_{\rho \in [c_1^*, c_2^*)} + L\mathbf{1}_{\rho \ge c_2^*}$$

is an optimal solution for Problem (cnp).

Optimal contingent with constraint

The optimal splitting $c^*, \tilde{x}_+^*$ can be determined by

$$
\text{(sp')} \quad
\begin{array}{ll}
\max & v_+(c, \tilde{x}_+) - v_3(c, c_2; c, \tilde{x}_+) \\
s.t. & \tilde{x}_+ \geq \tilde{x}_0, \operatorname{essinf} \rho \leq c < c_2 \leq \operatorname{esssup} \rho
\end{array}
$$

Optimal contingent with constraint

The optimal splitting $c^*, \tilde{x}_+^*$ can be determined by

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Theorem 8 Suppose Assumption 1 hold.

(i) If $(c^*, c_2^*, \tilde{x}_+^*)$ is optimal for Problem (sp'), then

$$X^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))})\mathbf{1}_{\rho \leq c^*} - q(c^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)\mathbf{1}_{\rho \in [c^*, c_2^*)} - L\mathbf{1}_{\rho \geq c_2^*} + B$$

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(ii) If there is no optimal $(c, c_2, \tilde{x}_+)$, then there is no optimal contingent claim.

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Theorem 9 If $h(x) = T_-(f^{-1}(x)))$ is a convex function, then the optimal splitting (c^*, c_2^*, x_+^*) satisfies $c^* = c_2^*$. Hence the optimal contingent claim is

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In any case, X^* is a two-piece function of ρ.

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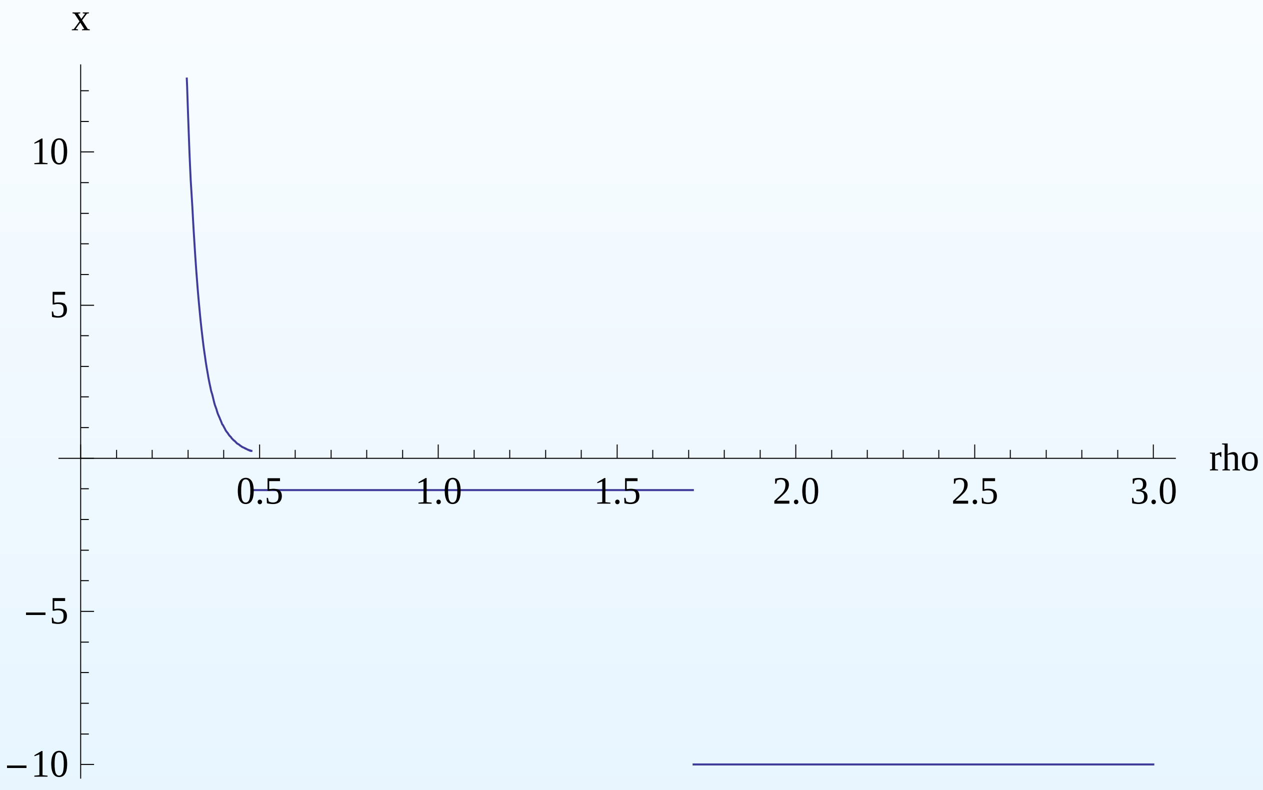
- An three-piece example:
  - $L = 10, \tilde{x}_0 = -1, \beta = 0.85, \alpha = 0.88, k = 2.25,$ $\rho \sim \text{Lognormal}(-0.045, 0.09)$
  - $h(x) =$
$$\begin{cases} 0.5x & x \in [0, 0.05] \\ 20 * 0.1^{\beta}(x - 0.05) + 0.025(0.1 - x) & x \in [0.05, 0.1] \\ x^{\beta} & x \in [0.1, 1] \end{cases}$$
  - The optimal solution $\tilde{X}^* = X^* - B$ is as in the next figure

Example: power value function



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- With loss control, the unconstrained optimal contingent claim is a three-piece function of ρ
- For power value function, the three-piece optimal contingent claim merge to two-piece function in many case

Thank you very much!