

# Portfolio Choice via Quantiles

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Based on the joint work with Prof Xunyu Zhou

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  - Kahneman and Tversky's prospect theory [Kahneman and Tversky (1979), Tversky and Kahneman (1992)]
  - Lopes' SP/A theory [Lopes (1987) and Lopes and Oden (1999)]

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  - Goal achieving problem [Kulldorff (1993), Heath (1993) and Browne (1999)]
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  - Law-invariant coherent risk measure [Artzner, Delbaen, Eber and Heath (1999) and Kusuoka (2001)]

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- Main difficulties include the non-concavity and time-inconsistency
- In this work, we propose a new framework to accommodate most of the aforementioned preferences and develop a new technique to solve the model

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- Dynamic portfolio selection can be translated into a static problem of choosing the optimal terminal payoff
- “The more money the better”

# A Non-Expected Utility Maximisation Model

- We consider the following portfolio selection problem

$$\begin{array}{ll} \text{Max}_X & V(X) := \int_{-\infty}^{\infty} u(x) d[-T(1 - F_X(x))] \\ \text{Subject to} & F_X(\cdot) \in \mathbb{F} \cap \mathbb{D}, \\ & E[\rho X] \leq x_0 \end{array}$$

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- $\mathbb{D}$  is a subset of  $\mathbb{F}$ , specifying the constraints imposed on the terminal payoff
- Both preference and constraints (other than the initial budget constraint) are *law-invariant*

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- Prospect Theory:

$$\int_B^{\infty} u_+(x - B) d[-T_+(1 - F_X(x))] - \int_{-\infty}^B u_-(B - x) d[T_-(F_X(x))]$$

# Change the Decision Variable

- The portfolio selection problem

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- The major difficulty comes from the distortion function
- Change of variable  $z = F_X(x)$

$$\begin{aligned} \int_{-\infty}^{\infty} u(x) d[-T(1 - F_X(x))] &= \int_0^1 u(F_X^{-1}(z)) T'(1 - z) dz \\ &= E[u(F_X^{-1}(Z)) T'(1 - Z)] \end{aligned}$$

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- If we regard  $F_X^{-1}(\cdot)$  as the variable, the distortion function is separated and we restore the concavity if  $u(\cdot)$  is concave

## Change the Decision Variable (Cont'd)

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- It also works in the constraints

$$F_X(\cdot) \in \mathbb{F} \cap \mathbb{D} \Leftrightarrow F_X^{-1}(\cdot) \in \mathbb{G} \cap \mathbb{M}$$

where

$$\mathbb{G} := \{G(\cdot) : (0, 1) \rightarrow \mathbb{R} \mid G(\cdot) \text{ is increasing, Càglàd and } G(0+) > -\infty\}$$

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- It, however, does not work in the budget constraint

$$E[\rho X] \leq x_0$$

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- The performance only depends on the distribution of the terminal payoff, thus the dual problem is to minimise the cost of replicating the terminal payoffs following a given distribution
- Given a distribution function  $F(\cdot)$ , formulate the following dual problem

$$\begin{array}{ll} \text{Min} & E[\rho X] \\ \text{Subject to} & X \text{ is } F(\cdot) \text{ distributed} \end{array}$$

## Lemma (Jin and Zhou 2008)

*If  $\rho$  has no atom, then  $Z := 1 - F_\rho(\rho)$  is uniformly distributed and  $E[\rho F^{-1}(Z)] \leq E[\rho X]$  for any  $F(\cdot)$  distributed r.v.  $X$ . Moreover, if  $E[\rho F^{-1}(Z)] < \infty$ , then the inequality is equality iff  $X = F^{-1}(Z)$*

- $X = F^{-1}(Z) = F^{-1}(1 - F_\rho(\rho))$ , where  $Z \sim U(0, 1)$ , uniquely solves the dual problem

# Terminal Payoff with Minimal Cost

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- This dual problem dates back to Dybvig (1988) and is revived in Jin and Zhou (2008)

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- Rewrite the budget constraint

$$\begin{aligned}x &\geq E [\rho F^{-1}(Z)] \\&= E [F_\rho^{-1}(1 - Z) F^{-1}(Z)] \quad (F_\rho^{-1}(F_\rho(\rho)) = \rho, \text{ a.s.}) \\&= \int_0^1 F_\rho^{-1}(1 - z) F^{-1}(z) dz\end{aligned}$$

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- We call it *quantile formulation*

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- All the aforementioned examples can be solved explicitly

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- The problem can be solved by Lagrange
- Incomplete market case can also be dealt with and mutual fund theorem is derived

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- SP/A model and model with law-invariant coherent risk measure have been solved by He and Zhou recently