A dual approach to some multiple exercise option problems

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Contents

American options

American option - option, which exercise is allowed at any time prior to the expiration date.

Problem formulation:

$$V_t^* = \sup_{t \le \tau \le T} E_t \left[B_t \frac{h_\tau}{B_\tau} \right], \tag{1}$$

h_t is the payoff from exersizing at time t.

B_t is the discount factor.

$\frac{h_\tau}{B_\tau}$ is the payoff discounted to time zero.

Single stopping

The optimal stopping formulation is equivalent to the dynamic programming equations

$$V_T^*(X_T) = h_T(X_T), \tag{2}$$

$$V_t^*(X_t) = \max\left\{h_t(X_t), E_t\left[\frac{B_t}{B_{t+1}}V_{t+1}^*(X_{t+1})\right]\right\} \tag{3}$$

The continuation value $C_t^*(X_t)$ is defined by

$$C_t^*(X_t) = E_t\left[\frac{B_t}{B_{t+1}}V_{t+1}^*(X_{t+1})\right], \qquad t = 0, 1, ..., T-1. \tag{4}$$

X_t are the state variables.

Longstaff-Schwartz algorithm

- The difficulty in the general problem is in estimating the continuation value.
- The Longstaff-Schwartz algorithm is a Monte Carlo method, which relies on least square regression of the continuation values from the simulated paths.
- The fitted value from this regression then gives an estimate for the continuation value.
- By estimating the continuation value an exercise rule is determined.
- The stopping rule gives a lower bound for the option price.

Longstaff-Schwartz algorithm

For all times $t \in \{0, 1, 2, ..., T\}$, at each point of the space set define an approximation to the continuation value by

$$\hat{C}_t(x) = \sum_{i=1}^{k} c_{t,i}\psi_i(x). \tag{5}$$

Let $\psi = (\psi_1, \psi_2, ..., \psi_k)$ and $\bar{c}_t = (c_{t,1}, c_{t,2}, ..., c_{t,k})$. If n paths are simulated, an estimation for the regression coefficients would be

$$\arg\min_{c \in R^k} \sum_{j=1}^{n} \left(C_t^{(j)} - \sum_{i=1}^{k} c_{t,i}\psi_i(X_t^{(j)})\right)^2, \tag{6}$$

where

$$C_t^{(j)} = \frac{B_t}{B_{t+1}} V_{t+1}^{(j)}. \tag{7}$$

Dual approach

- The method relies on a dual represenaion of the value function.
- The problem becomes equivalent to minimization of the dual representation over a set of martingales.
- The optimal martingale (the one that achieves the infimum) is known.
- The problem comes down to approximating the optimal martingale.

Dual approach

V_t^*/B_t is a supermartingale

$$E_t\left[\frac{B_t}{B_{t+1}} V_{t+1}^*(X_{t+1})\right] \leq V_t^*(X_t), \tag{8}$$

Here from the Doob-Meyer decomposition

$$\frac{V_t^*}{B_t} = V_0^* + M_t^* - D_t^*,$$

where M_t^* is a martingale and D_t^* is an increasing process, both vanishing at $t = 0$.
Theorem(Rogers; Haugh and Kogan) The value function V_0^* at time zero is given by

$$V_0^* = \inf_{M \in H_0^1} E[\sup_{0 \leq t \leq T} (\frac{h_t}{B_t} - M_t)], \tag{9}$$

where H_0^1 is the space of martingales M, for which $\sup_{0 \leq t \leq T} |M_t|$ is integrable and such that $M_0 = 0$. The infimum is attained by taking $M = M^*$. The optimal martingale here can be expressed by $M_{t+1}^* - M_t^* = \frac{V_{t+1}^*}{B_{t+1}} - E_t[\frac{V_{t+1}^*}{B_{t+1}}]$.

Multiple Stopping - Motivation

The motivation for these optimal stopping problems comes from pricing swing contracts with the following features:

1 The swing option has maturity T days and can be exercised on days $1, 2, ..., T$.

2 It can be exercised up to k_t times on day t and the total number of exercise rights is m .

3 When exercising the option, its holder buys a certain number of units (usually 1MWh) of electricity for a prespecified fixed price K.

Multiple stopping

We define an exercise policy π to be a set of stopping times $\{\tau_i\}_{i=1}^m$ with $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_m$ and $\#\{j : \tau_j = s\} \leq k_s$. Then the value of the policy π at time t is given by

$$V_t^{\pi,m,\mathbf{k}} = \mathbb{E}_t(\sum_{i=1}^{m} h_{\tau_i}(X_{\tau_i})).$$

The value function is defined to be

$$V_t^{*,m,\mathbf{k}} = \sup_{\pi} V_t^{\pi,m,\mathbf{k}} = \sup_{\pi} \mathbb{E}_t(\sum_{i=1}^{m} h(X_{\tau_i})).$$

We denote the corresponding optimal policy $\pi^* = \{\tau_1^*, \tau_2^*, ..., \tau_m^*\}$.

Multiple stopping

(Multiple exercise option price - Dynamic programming formulation) The price $V_t^{*,m,\mathbf{k}}$ at time t of an option with payoff function $\{h_s, t \leq s \leq T\}$ which could be exercised k_s times per single exercise time $s \in \{t, \ldots, T\}$ with m exercise opportunities in total for $m > k_t$ is given by

$$V_T^{*,m,\mathbf{k}} = k_T h_T,$$

$$V_t^{*,m,\mathbf{k}} = \max\{k_t h_t + \mathbb{E}_t[V_{t+1}^{*,m-k_t,\mathbf{k}}], (k_t - 1)h_t + \mathbb{E}_t[V_{t+1}^{*,m-(k_t-1),\mathbf{k}}], \ldots, h_t + \mathbb{E}_t[V_{t+1}^{*,m-1,\mathbf{k}}], \mathbb{E}_t[V_{t+1}^{*,m,\mathbf{k}}]\}.$$

For $m \leq k_t$ we have

$$V_T^{*,m,\mathbf{k}} = m h_T,$$

$$V_t^{*,m,\mathbf{k}} = \max\{m h_t, (m-1)h_t + \mathbb{E}_t[V_{t+1}^{*,1,\mathbf{k}}], \ldots, \mathbb{E}_t[V_{t+1}^{*,m,\mathbf{k}}]\}.$$

Multiple stopping

(Multiple exercise option price - Optimal stopping problem formulation) The price $V_t^{*,m,\mathbf{k}}$ of an option, which could be exercised k_t times per single exercise time t with m exercise opportunities in total is given by

$$V_t^{*,m,\mathbf{k}} = \max_{t \leq \tau \leq T} \mathbb{E}_t \Big[\max\{k_\tau h_\tau + \mathbb{E}_\tau[V_{\tau+1}^{*,m-k_\tau,\mathbf{k}}], (k_\tau - 1)h_\tau + \mathbb{E}_\tau[V_{\tau+1}^{*,m-(k_\tau-1),\mathbf{k}}], \\ \ldots, h_\tau + \mathbb{E}_\tau[V_{\tau+1}^{*,m-1,\mathbf{k}}]\}\Big]$$

(In the $\max$ bracket only those terms, which exist are taken)

Marginal value The marginal value of one additional exercise opportunity is denoted by $\Delta V_t^{*,m,\mathbf{k}}$ for $m \geq 1$:

$$\Delta V_t^{*,m,\mathbf{k}} = V_t^{*,m,\mathbf{k}} - V_t^{*,m-1,\mathbf{k}}.$$

The marginal value for $m = 1$ is just the option value for one exercise opportunity

$$\Delta V_t^{*,1,\mathbf{k}} = V_t^{*,1,\mathbf{k}}.$$

Multiple stopping - lower

Generalisation of Longstaff-Schwartz.

Suppose that, working backwards in time and forward from one exercise opportunity, approximations $\Delta\hat{C}^{m}_{t+1}, \Delta\hat{C}^{m-1}_{t+1}, \ldots, \Delta\hat{C}^{m-k_{t+1}+1}_{t+1}$ to the $m-$th, $m-1$,...,$m-k_{t+1}+1$ marginal continuation value functions have been obtained. Then for path j define the approximate continuation value $C^{m,(j)}_{t}$ to be

$$C^{m,(j)}_{t} = \begin{cases} k_{t+1}h_{t+1}(X^{(j)}_{t+1}) + C^{m-k_{t+1},(j)}_{t+1}, \\ \qquad \text{if } h_{t+1}(X^{(j)}_{t+1}) \geq \Delta\hat{C}^{m-k_{t+1}+1}_{t+1}(X^{(j)}_{t+1}) \\ (k_{t+1}-1)h_{t+1}(X^{(j)}_{t+1}) + C^{m-k_{t+1}+1,(j)}_{t+1}, \\ \qquad \text{if } \Delta\hat{C}^{m-k_{t+1}+1}_{t+1}(X^{(j)}_{t+1}) > h_{t+1}(X^{(j)}_{t+1}) \geq \Delta\hat{C}^{m-k_{t+1}+2}_{t+1}(X^{(j)}_{t+1}) \\ \qquad\quad \vdots \\ C^{m,(j)}_{t+1}, \\ \qquad \text{if } \Delta\hat{C}^{m}_{t+1}(X^{(j)}_{t+1}) > h_{t+1}(X^{(j)}_{t+1}) \end{cases}$$

The non-optimal $m-$th marginal continuation values are also defined by
$\Delta C^{m,(j)}_{t} = C^{m,(j)}_{t} - C^{m-1,(j)}_{t}$.

Multiple stopping - upper

Theorem(Aleksandrov and Hambly; Bender) The marginal value $\Delta V_0^{*,m,\mathbf{k}}$ is equal to

$$\Delta V_0^{*,m,\mathbf{k}} = \inf_{\pi} \inf_{\mathcal{M} \in H_0} E_0 \Big[\max_{u \in (G_0 \setminus \{\tau_{m-1},\ldots,\tau_1\})} (h_u - \mathcal{M}_u) \Big],$$

where the infima are taken over all stopping policies π and over the set of integrable martingales H_0.

- We define $\bar{N}_t(\tau_m, \ldots, \tau_1)$ to be the number of stopping time in the multiset $\tau_m, \ldots, \tau_1$ that are less than or equal to t.
- The optimal martingale is defined by

$$\mathcal{M}^*_{t+1} - \mathcal{M}^*_t = \sum_{l=0}^{m-1} (\Delta M_{t+1}^{*,m-l,\mathbf{k}} - \Delta M_t^{*,m-l,\mathbf{k}}) \mathbf{1}_{\bar{N}^*_t = l}$$

- The optimal stopping policy π here is the optimal stopping policy for the problem with $m-1$ exercise rights.
- G_0 is the multiset of all possible stopping times.

Multiple stopping - upper

- Property 1.

$$\Delta V_t^{*,m+1,\mathbf{k}} \leq \Delta V_t^{*,m,\mathbf{k}}, \qquad \forall t.$$

- Property 2.

$$\Delta V_t^{*,m,\mathbf{k}} = \max_{t \leq \tau \leq T} \mathbb{E}_t \left[\min(h_\tau, \mathbb{E}_\tau[\Delta V_{\tau+1}^{*,m-k_\tau,\mathbf{k}}]) - D_\tau^{*,m-1,\mathbf{k}} \right] + D_t^{*,m-1,\mathbf{k}}.$$

- Property 3.

$$\Delta V_0^{*,m,\mathbf{k}} = \inf_{0 \leq \tau \leq T} \inf_{M \in H_0} \mathbb{E}_0 \Big[\max_{0 \leq t \leq \tau} \Big(\min(h_t, \mathbb{E}_t[\Delta V_{t+1}^{*,m-k_t,\mathbf{k}}]) \mathbf{1}_{t<\tau} + \max(\min(h_t, \mathbb{E}_t[\Delta V_{t+1}^{*,m-k_t,\mathbf{k}}]), \mathbb{E}_t[\Delta V_{t+1}^{*,m-1,\mathbf{k}}]) \mathbf{1}_{t=\tau} - M_t \Big) \Big],$$

where the infima are taken over all stopping times τ and over the set of integrable martingales H_0. The infimum is attained for the martingale $\Delta M_t^{*,m,\mathbf{k}}$ and stopping time τ defined as $\tau^* = \min\{t : D_{t+1}^{*,m-1,\mathbf{k}} > 0\}$.

Numerical Example

We consider a electricity swing option with a lifetime $T = 1000$. The option can be exercised once on a weekend and twice on a weekday.
The underlying process (electricity spot price) is the exponential of a discrete mean reverting process

$$\log S_{t+1} = (1 - \alpha) \log S_t + \sigma W_t. \tag{10}$$

With parameters $S_0 = 1, \sigma = 0.5$ and $\alpha = 0.9$. The payoff is taken to be the spot price itself.
The basis functions used for approximating the marginal continuation values are

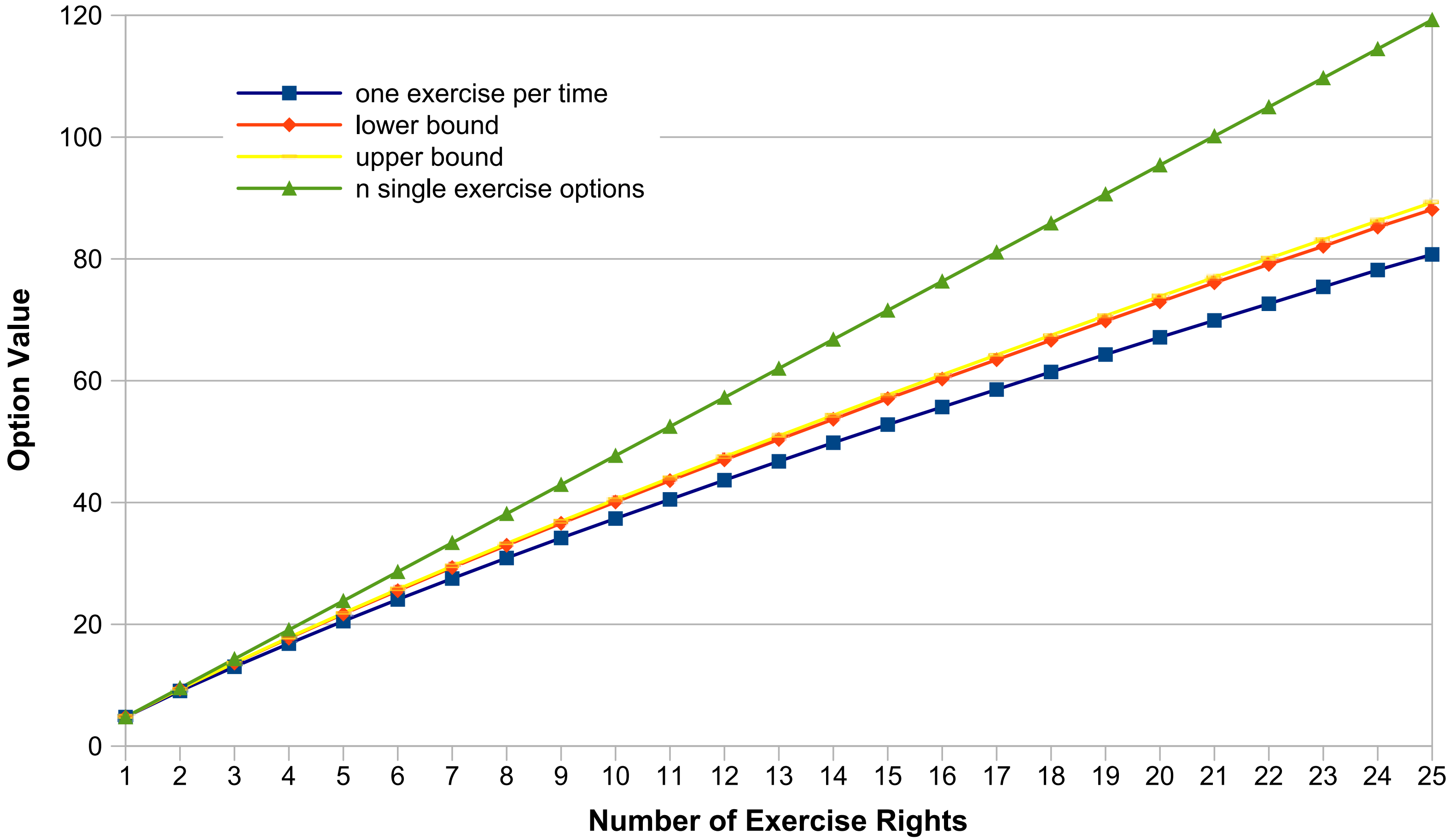
$$\Psi = \{1, \log S\}. \tag{11}$$

We use 10000 pre-simulation path to determine the stopping strategy and 20000 paths for the lower bound, given the stopping strategy. For the upper bound we use 1000 paths and 50 inner paths for the martingale approximation.

Numerical Results

Exercise possibilities	1 ex. right per time	lower	upper	standard dev upper bound	relative difference	upper bound marginal value
1	4.77	4.77	4.79	0.21	0.004	4.79
2	9.06	9.37	9.39	0.29	0.002	4.60
3	13.03	13.65	13.68	0.34	0.002	4.29
4	16.83	17.73	17.83	0.38	0.006	4.15
5	20.52	21.70	21.84	0.41	0.006	4.01
6	24.08	25.52	25.74	0.44	0.008	3.90
7	27.52	29.32	29.54	0.46	0.008	3.80
8	30.89	32.98	33.26	0.48	0.008	3.72
9	34.18	36.59	36.91	0.50	0.009	3.65
10	37.37	40.08	40.50	0.52	0.010	3.59
15	52.80	57.05	57.66	0.59	0.011	3.34
20	67.13	72.95	73.83	0.64	0.012	3.17
25	80.74	88.12	89.27	0.68	0.013	3.04

Numerical Results



Advantages & Disadvantages

Advantages:

- The approach is model independent.
- Provides upper and lower bounds for the value function.
- Monte Carlo is linear in the dimensionality.

Disadvantages:

- Basis functions are in some cases hard to choose.

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Questions

Thank you for your attention.

Questions?