Volatility and Liquidity Trading

René Carmona and Z. Joseph Yang

Bendheim Center for Finance & Dept. of ORFE, Princeton University

March 28th, 2009
Outline

1 Motivation

2 Market Impact Modeling
   - The Permanent Component is Necessarily Linear
   - Time-Homogeneity is Obtained By Subordinating in Volume-Time

3 Formulation of the Liquidity Trading Game
   - The Stochastic Optimal Control Problem for Each Player
   - The Nash-Equilibrium for the Liquidity Trading Game
   - Numerical Analysis of the NE

4 Summary and Ongoing Work
The Liquidity Nature of the Financial Markets

- **Liquidity Risk**
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - **Funding Liquidity**
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- **Market Risk**
- **Credit Risk**
The Liquidity Nature of the Financial Markets

- Liquidity Risk
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- Market Risk
- Credit Risk
The Liquidity Nature of the Financial Markets

- **Liquidity Risk**
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- Market Risk
- Credit Risk
The Liquidity Nature of the Financial Markets

- Liquidity Risk
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- Market Risk
- Credit Risk
The Liquidity Nature of the Financial Markets

Liquidity Risk
  Asset Liquidity / Market Liquidity
  inventory approach/information-asymmetry approach
  empirical data modeling approach
  optimal control theory approach
  game theoretic approach

Funding Liquidity
  margin requirement shift and illiquidity spiral
  capital structure analysis
  etc.

Market Risk
Credit Risk
The Liquidity Nature of the Financial Markets

- Liquidity Risk
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- Market Risk
- Credit Risk
The Liquidity Nature of the Financial Markets

- Liquidity Risk
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- Market Risk
- Credit Risk
The Liquidity Nature of the Financial Markets

- Liquidity Risk
  - Asset Liquidity / Market Liquidity
    - inventory approach/information-asymmetry approach
    - empirical data modeling approach
    - optimal control theory approach
    - game theoretic approach
  - Funding Liquidity
    - margin requirement shift and illiquidity spiral
    - capital structure analysis
    - etc.

- Market Risk
- Credit Risk
The crucial role of understanding the Liquidity Nature of a financial market, for both market participants and regulators alike

- Black Monday in 1987
- LTCM and sovereign bond crisis in 1998
- Riot of subprime credit products in 2007
- the collapse of Amaranth in 2006, and so on
The crucial role of understanding the Liquidity Nature of a financial market, for both market participants and regulators alike

- Black Monday in 1987
- LTCM and sovereign bond crisis in 1998
- Riot of subprime credit products in 2007
- the collapse of Amaranth in 2006, and so on
The crucial role of understanding the Liquidity Nature of a financial market, for both market participants and regulators alike

- Black Monday in 1987
- LTCM and sovereign bond crisis in 1998
- Riot of subprime credit products in 2007
- the collapse of Amaranth in 2006, and so on
The crucial role of understanding the Liquidity Nature of a financial market, for both market participants and regulators alike

- Black Monday in 1987
- LTCM and sovereign bond crisis in 1998
- Riot of subprime credit products in 2007
- the collapse of Amaranth in 2006, and so on
The crucial role of understanding the Liquidity Nature of a financial market, for both market participants and regulators alike

- Black Monday in 1987
- LTCM and sovereign bond crisis in 1998
- Riot of subprime credit products in 2007
- the collapse of Amaranth in 2006, and so on
Intension/activity of buying and selling does affect the market price of an asset
Trading tactics inspired by the liquidity rationale and trend of the market

- optimal execution for a single player
  Bertsimas & Lo, 98; Almgren & Chriss, 01; Almgren 03
- strategic play between multiple players
  Brunnermeier & Pedersen, 05
  Carlin, Lobo, Viswanathan, 07
  Schoneborn & Schied, 07
Intension/activity of buying and selling does affect the market price of an asset. Trading tactics inspired by the liquidity rationale and trend of the market:

- **optimal execution for a single player**
  - Bertsimas & Lo, 98; Almgren & Chriss, 01; Almgren 03
- **strategic play between multiple players**
  - Brunnermeier & Pedersen, 05
  - Carlin, Lobo, Viswanathan, 07
  - Schoneborn & Schied, 07
Intension/activity of buying and selling does affect the market price of an asset.

Trading tactics inspired by the liquidity rationale and trend of the market:

- **optimal execution for a single player**
  - Bertsimas & Lo, 98; Almgren & Chriss, 01; Almgren 03

- **strategic play between multiple players**
  - Brunnermeier & Pedersen, 05
  - Carlin, Lobo, Viswanathan, 07
  - Schoneborn & Schied, 07
Intension/activity of buying and selling does affect the market price of an asset.

Trading tactics inspired by the liquidity rationale and trend of the market:

- optimal execution for a single player
  Bertsimas & Lo, 98; Almgren & Chriss, 01; Almgren 03
- strategic play between multiple players
  - Brunnermeier & Pedersen, 05
  - Carlin, Lobo, Viswanathan, 07
  - Schoneborn & Schied, 07
Intension/activity of buying and selling does affect the market price of an asset
Trading tactics inspired by the liquidity rationale and trend of the market

- optimal execution for a single player
  Bertsimas & Lo, 98; Almgren & Chriss, 01; Almgren 03
- strategic play between multiple players
  - Brunnermeier & Pedersen, 05
  - Carlin, Lobo, Viswanathan, 07
  - Schoneborn & Schied, 07
A quick overview of previous models

- Trading in continuous-time → differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role
A quick overview of previous models

- Trading in continuous-time → differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role
A quick overview of previous models

- Trading in continuous-time $\rightarrow$ differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role
A quick overview of previous models

- Trading in continuous-time → differential game
- Permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- Only open-loop strategies are allowed for each player
- Essentially deterministic optimal control and volatility never plays a role
A quick overview of previous models

- Trading in continuous-time $\rightarrow$ differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role
A quick overview of previous models

- Trading in continuous-time $\rightarrow$ differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role
A quick overview of previous models

- Trading in continuous-time → differential game
- permanent component and temporary component of market impact
- Nash-equilibrium of the game, either predation or providing liquidity
- only open-loop strategies are allowed for each player
- essentially deterministic optimal control and volatility never plays a role
Market Impact Modeling

\[ X(t) = \int_{0}^{t} \xi(s) \, ds, \]  
where \( \xi(t) \) is the trading intensity in continuous-time.

The market (mid-quote) price is given by

\[ dP(t) = \mu(t, \cdots)dt + f(\xi(t))dt + \sigma(t, \cdots)dW(t) \quad (1) \]

\[ \text{COST} = \int_{0}^{t} \tilde{P}(s)\xi(s) \, ds = \int_{0}^{t} (P(s) + g(\xi(s)))\xi(s) \, ds \]

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Market Impact Modeling

\[ X(t) = \int_0^t \xi(s) \, ds, \text{ where } \xi(t) \text{ is the trading intensity in continuous-time.} \]

market (mid-quote) price

\[
dP(t) = \mu(t, \cdots) \, dt + f(\xi(t)) \, dt + \sigma(t, \cdots) \, dW(t) \tag{1}
\]

\[
COST = \int_0^t \tilde{P}(s) \xi(s) \, ds = \int_0^t (P(s) + g(\xi(s))) \xi(s) \, ds
\]

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Market Impact Modeling

\( X(t) = \int_0^t \xi(s) \, ds \), where \( \xi(t) \) is the trading intensity in continuous-time.

market (mid-quote) price

\[
dP(t) = \mu(t, \cdots) \, dt + f(\xi(t)) \, dt + \sigma(t, \cdots) \, dW(t)
\]  

\( COST = \int_0^t \tilde{P}(s) \xi(s) \, ds = \int_0^t (P(s) + g(\xi(s)))\xi(s) \, ds \)

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Market Impact Modeling

\[ X(t) = \int_0^t \xi(s) \, ds, \text{ where } \xi(t) \text{ is the trading intensity in continuous-time.} \]

market (mid-quote) price

\[ dP(t) = \mu(t, \cdots) \, dt + f(\xi(t)) \, dt + \sigma(t, \cdots) \, dW(t) \quad (1) \]

\[ COST = \int_0^t \tilde{P}(s) \xi(s) \, ds = \int_0^t (P(s) + g(\xi(s))) \xi(s) \, ds \]

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Market Impact Modeling

\[ X(t) = \int_0^t \xi(s) \, ds, \] where \( \xi(t) \) is the trading intensity in continuous-time.

market (mid-quote) price

\[ dP(t) = \mu(t, \cdots) \, dt + f(\xi(t)) \, dt + \sigma(t, \cdots) \, dW(t) \quad (1) \]

\[ COST = \int_0^t \tilde{P}(s) \xi(s) \, ds = \int_0^t (P(s) + g(\xi(s))) \xi(s) \, ds \]

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Market Impact Modeling

\[ X(t) = \int_0^t \xi(s) \, ds, \] where \( \xi(t) \) is the trading intensity in continuous-time.

market (mid-quote) price

\[
dP(t) = \mu(t, \cdots) \, dt + f(\xi(t)) \, dt + \sigma(t, \cdots) \, dW(t) \tag{1}
\]

\[
COST = \int_0^t \tilde{P}(s) \xi(s) \, ds = \int_0^t (P(s) + g(\xi(s))) \xi(s) \, ds
\]

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Market Impact Modeling

\[ X(t) = \int_0^t \xi(s) \, ds, \]  
where \( \xi(t) \) is the trading intensity in continuous-time.

market (mid-quote) price

\[
dP(t) = \mu(t, \ldots) \, dt + f(\xi(t)) \, dt + \sigma(t, \ldots) \, dW(t) \tag{1}
\]

\[
COST = \int_0^t \tilde{P}(s) \xi(s) \, ds = \int_0^t (P(s) + g(\xi(s))) \xi(s) \, ds
\]

where \( f(\cdot) \) and \( g(\cdot) \) are the so-called permanent component function and temporary component function.
Call a trading scheme a *clean-hand trade* if the strategy \( \{\xi(t)\}_{t \in [0,T]} \) satisfies \( 0 = \int_0^T \xi(t) \, dt = X(T) - X(0) \), and the integral process \( X(t) \) is bounded.

\[
\Pi = \mathbb{E} \left[ - \int_0^T \tilde{P}(t) \xi(t) \, dt \right] \leq 0
\]
Call a trading scheme a *clean-hand trade* if the strategy 
\[ \{\xi(t)\}_{t \in [0,T]} \] satisfies 
\[ 0 = \int_0^T \xi(t) \, dt = X(T) - X(0), \] 
and the integral process \( X(t) \) is bounded.

\[
\Pi = \mathbb{E} \left[ - \int_0^T \tilde{P}(t) \xi(t) \, dt \right] \leq 0
\] (2)
Call a trading scheme a *clean-hand trade* if the strategy \( \{ \xi(t) \}_{t \in [0, T]} \) satisfies \( 0 = \int_0^T \xi(t) \, dt = X(T) - X(0) \), and the integral process \( X(t) \) is bounded.

\[
\Pi = \mathbb{E} \left[ - \int_0^T \tilde{P}(t) \xi(t) \, dt \right] \leq 0 \tag{2}
\]
Market Impact Modeling

\[ \Pi = \mathbb{E} \left[ - \int_0^T (P(t) + g(\xi(t)))\xi(t) \, dt \right] \]  

\[ = \mathbb{E} \left[ - \int_0^T P(t) \, dX(t) - \int_0^T g(\xi(t))\xi(t) \, dt \right] \]  

\[ = \mathbb{E} \left[ - P(t)X(t) \bigg|_0^T + \int_0^T X(t) \left( f(\xi(t)) \, dt + \sigma(t, P(t)) \, dW(t) \right) \right. \]  

\[ - \int_0^T g(\xi(t))\xi(t) \, dt \right] \]  

\[ = \int_0^T X(t)f(\xi(t)) \, dt - \int_0^T g(\xi(t))\xi(t) \, dt \]  

\[ + \mathbb{E} \left[ \int_0^T X(t)\sigma(t, P(t)) \, dW(t) \right] \]
Market Impact Modeling

\[
\Pi = \mathbb{E} \left[ -\int_0^T (P(t) + g(\xi(t)))\xi(t) \, dt \right]
\]

\[
= \mathbb{E} \left[ -\int_0^T P(t) \, dX(t) - \int_0^T g(\xi(t))\xi(t) \, dt \right]
\]

\[
= \mathbb{E} \left[ -P(t)X(t)|_0^T + \int_0^T X(t) \left( f(\xi(t)) \, dt + \sigma(t, P(t)) \, dW(t) \right) \right]
\]

\[
- \int_0^T g(\xi(t))\xi(t) \, dt
\]

\[
= \int_0^T X(t)f(\xi(t)) \, dt - \int_0^T g(\xi(t))\xi(t) \, dt
\]

\[
+ \mathbb{E} \left[ \int_0^T X(t)\sigma(t, P(t)) \, dW(t) \right]
\]
Market Impact Modeling

\[ \Pi = \int_0^{\frac{T}{2}} \xi t f(\xi) \, dt + \int_{\frac{T}{2}}^{T} \xi (T - t) f(-\xi) \, dt \]
\[ - \int_0^{\frac{T}{2}} g(\xi) \xi \, dt - \int_{\frac{T}{2}}^{T} g(-\xi)(-\xi) \, dt \]
\[ = \frac{T^2}{8} \xi (f(\xi) + f(-\xi)) + \frac{T}{2} \xi (g(-\xi) - g(\xi)) \]
\[ \leq 0 \]

\[ \Rightarrow f(-\xi) = -f(\xi), \text{ for any } \xi \in \mathbb{R} \]
and \[ g(\xi) \geq g(-\xi), \text{ for any } \xi > 0 \]
Volatility and Liquidity Trading

Motivation

Market Impact Modeling

Permanent Component is Necessarily Linear
Subordinating in Volume-Time

Liquidity Trading Game

Summary

References

Market Impact Modeling

\[
\Pi = \int_0^{T/2} \xi tf(\xi) \, dt + \int_{T/2}^T \xi(T - t)f(-\xi) \, dt
\]

\[
- \int_0^{T/2} g(\xi)\xi \, dt - \int_{T/2}^T g(-\xi)(-\xi) \, dt
\]

\[
= \frac{T^2}{8}\xi (f(\xi) + f(-\xi)) + \frac{T}{2}\xi (g(-\xi) - g(\xi))
\]

\[
\leq 0
\]

\Rightarrow f(-\xi) = -f(\xi), \text{ for any } \xi \in \mathbb{R}

and \( g(\xi) \geq g(-\xi), \text{ for any } \xi > 0 \)
Market Impact Modeling

\[ \Pi = \int_0^{T/2} \xi t f(\xi) \, dt + \int_{T/2}^{T} \xi (T - t) f(-\xi) \, dt \]

\[ - \int_0^{T/2} g(\xi) \xi \, dt - \int_{T/2}^{T} g(-\xi) (-\xi) \, dt \]

\[ = \frac{T^2}{8} \xi \left( f(\xi) + f(-\xi) \right) + \frac{T}{2} \xi \left( g(-\xi) - g(\xi) \right) \leq 0 \]

\[ \Rightarrow f(-\xi) = -f(\xi), \text{ for any } \xi \in \mathbb{R} \]

and \[ g(\xi) \geq g(-\xi), \text{ for any } \xi > 0 \]
\[ \Pi = \int_{0}^{\frac{\xi_2}{\xi_1 + \xi_2} T} \xi_1 t f(\xi_1) \, dt + \int_{\frac{\xi_2}{\xi_1 + \xi_2} T}^{T} \xi_2 (T - t) f(-\xi_2) \, dt \\
- \int_{0}^{\frac{\xi_2}{\xi_1 + \xi_2} T} g(\xi_1) \xi_1 \, dt - \int_{\frac{\xi_2}{\xi_1 + \xi_2} T}^{T} g(-\xi_2) (-\xi_2) \, dt \\
= \xi_1 f(\xi_1) \frac{1}{2} \cdot \frac{\xi_2^2 T^2}{(\xi_1 + \xi_2)^2} + \xi_2 f(-\xi_2) \frac{1}{2} \cdot \frac{\xi_1^2 T^2}{(\xi_1 + \xi_2)^2} \\
- g(\xi_1) \xi_1 \frac{\xi_2 T}{\xi_1 + \xi_2} + g(-\xi_2) \xi_2 \frac{\xi_1 T}{\xi_1 + \xi_2} \\
= \frac{T^2}{2} \frac{\xi_1 \xi_2}{(\xi_1 + \xi_2)^2} (\xi_2 f(\xi_1) - \xi_1 f(\xi_2)) + T \frac{\xi_1 \xi_2}{\xi_1 + \xi_2} (g(-\xi_2) - g(\xi_1)) \\
\leq 0 \\
\Rightarrow \xi_2 f(\xi_1) - \xi_1 f(\xi_2) = 0, \text{ for any } \xi_1, \xi_2 \in \mathbb{R} \\
\text{necessarily, there } \exists \gamma, \text{ s.t. } f(\xi) = \gamma \xi, \text{ for any } \xi \in \mathbb{R} \]
\[\Pi = \int_0^{\frac{\xi_2}{\xi_1+\xi_2}T} \xi_1 t f(\xi_1) \, dt + \int_{\frac{\xi_2}{\xi_1+\xi_2}T}^T \xi_2 (T-t) f(-\xi_2) \, dt \]

\[-\int_0^{\frac{\xi_2}{\xi_1+\xi_2}T} g(\xi_1) \xi_1 \, dt - \int_{\frac{\xi_2}{\xi_1+\xi_2}T}^T g(-\xi_2)(-\xi_2) \, dt\]

\[= \xi_1 f(\xi_1) \frac{1}{2} \cdot \frac{\xi_2^2 T^2}{(\xi_1 + \xi_2)^2} + \xi_2 f(-\xi_2) \frac{1}{2} \cdot \frac{\xi_1^2 T^2}{(\xi_1 + \xi_2)^2} \]

\[-g(\xi_1)\xi_1 \frac{\xi_2 T}{\xi_1 + \xi_2} + g(-\xi_2)\xi_2 \frac{\xi_1 T}{\xi_1 + \xi_2}\]

\[= \frac{T^2}{2} \frac{\xi_1 \xi_2}{(\xi_1 + \xi_2)^2} (\xi_2 f(\xi_1) - \xi_1 f(\xi_2)) + T \frac{\xi_1 \xi_2}{\xi_1 + \xi_2} (g(-\xi_2) - g(\xi_1)) \leq 0\]

\[\Rightarrow \xi_2 f(\xi_1) - \xi_1 f(\xi_2) = 0, \text{ for any } \xi_1, \xi_2 \in \mathbb{R}\]

namely, there \(\exists \gamma, \text{ s.t. } f(\xi) = \gamma \xi, \text{ for any } \xi \in \mathbb{R}\)
This analytical argument is supported by empirical studies such as Almgren et al. (05) conducted on large-scale datasets traded at NYSE.

Time-homogeneity can be obtained by rescaling real time using the intra-day volume up to that moment, so-called *volume time*.

E.g., a VWAP execution in real time is essentially a constant-intensity trading trajectory in volume time.
This analytical argument is supported by empirical studies such as Almgren et al. (05) conducted on large-scale datasets traded at NYSE.

Time-homogeneity can be obtained by rescaling real time using the intra-day volume up to that moment, so-called *volume time*.

E.g., a VWAP execution in real time is essentially a constant-intensity trading trajectory in volume time.
The Story of a Liquidity Trading Game

- A distressed trader with maximum inventory $x_0$, constrained by an exogenous time horizon $[0, T]$
  - Objective, to generate cash as much as possible
  - Only allowed to monotonely sell (liquidate), not allowed to buy back at any moment during $[0, T]$

- A perfectly solvent trader, sophisticated and aggressive
  - Able to buy or sell at any moment
  - The only constraint is to be clean-hand by $\bar{T} \gg T$.

- Each player looks to closed-loop optimal control strategies, aiming to utilize the updates/feedback of market evolution to refine her control.
  - $\Rightarrow$ Subgame-Perfect.
The Story of a Liquidity Trading Game

- A distressed trader with maximum inventory $x_0$, constrained by an exogenous time horizon $[0, T]$
  - Objective, to generate cash as much as possible
  - Only allowed to monotonely sell (liquidate), not allowed to buy back at any moment during $[0, T]$
- A perfectly solvent trader, sophisticated and aggressive
  - Able to buy or sell at any moment
  - The only constraint is to be *clean-hand* by $\bar{T} \gg T$.
- Each player looks to closed-loop optimal control strategies, aiming to utilize the updates/feedback of market evolution to refine her control.
  - $\Rightarrow$ Subgame-Perfect.
The Story of a Liquidity Trading Game

- A distressed trader with maximum inventory $x_0$, constrained by an exogenous time horizon $[0, T]$
  - Objective, to generate cash as much as possible
  - Only allowed to monotonely sell (liquidate), not allowed to buy back at any moment during $[0, T]$
- A perfectly solvent trader, sophisticated and aggressive
  - Able to buy or sell at any moment
  - The only constraint is to be *clean-hand* by $\bar{T} \gg T$.
- Each player looks to closed-loop optimal control strategies, aiming to utilize the updates/feedback of market evolution to refine her control.
  - $\Rightarrow$ Subgame-Perfect.
Setting up the Liquidity Trading Game Model

An illiquid asset with permanent component coef $\gamma$, temporary component coef $\lambda$, intra-day volatility $\sigma$, in common knowledge to the two players

$$dZ(t) = (\gamma \xi(t) + \gamma \eta(t))dt + \sigma dW(t)$$

CL control strategies for the two players

$$\phi(t, x, y, z) \quad \text{and} \quad \psi(t, x, y, z)$$

Given the CL strategy of the opponent, each player solves her optimal control problem

Agreeing at the Nash-equilibrium of this game, when nobody has incentive to deviate.
An illiquid asset with permanent component coef $\gamma$, temporary component coef $\lambda$, intra-day volatility $\sigma$, in common knowledge to the two players

$$dZ(t) = (\gamma \xi(t) + \gamma \eta(t))dt + \sigma dW(t)$$

CL control strategies for the two players

$$\phi(t, x, y, z) \quad \text{and} \quad \psi(t, x, y, z)$$

Given the CL strategy of the opponent, each player solves her optimal control problem
Agreeing at the Nash-equilibrium of this game, when nobody has incentive to deviate.
Setting up the Liquidity Trading Game Model

An illiquid asset with permanent component coef $\gamma$, temporary component coef $\lambda$, intra-day volatility $\sigma$, in common knowledge to the two players

$$dZ(t) = (\gamma \xi(t) + \gamma \eta(t)) dt + \sigma dW(t)$$

CL control strategies for the two players

$$\phi(t, x, y, z) \quad \text{and} \quad \psi(t, x, y, z)$$

Given the CL strategy of the opponent, each player solves her optimal control problem

Agreeing at the Nash-equilibrium of this game, when nobody has incentive to deviate.
Setting up the Liquidity Trading Game Model

An illiquid asset with permanent component coefficient $\gamma$, temporary component coefficient $\lambda$, intra-day volatility $\sigma$, in common knowledge to the two players

$$dZ(t) = (\gamma \xi(t) + \gamma \eta(t))dt + \sigma dW(t)$$

CL control strategies for the two players

$$\phi(t, x, y, z) \quad \text{and} \quad \psi(t, x, y, z)$$

Given the CL strategy of the opponent, each player solves her optimal control problem

Agreeing at the Nash-equilibrium of this game, when nobody has incentive to deviate.
Setting up the Liquidity Trading Game Model

An illiquid asset with permanent component coef $\gamma$, temporary component coef $\lambda$, intra-day volatility $\sigma$, in common knowledge to the two players

$$dZ(t) = (\gamma \xi(t) + \gamma \eta(t))dt + \sigma dW(t)$$

CL control strategies for the two players

$$\phi(t, x, y, z) \quad \text{and} \quad \psi(t, x, y, z)$$

Given the CL strategy of the opponent, each player solves her optimal control problem

Agreeing at the Nash-equilibrium of this game, when nobody has incentive to deviate.
The Stochastic Optimal Control Problem

Given the CL strategy \( \psi(\cdots) \) of the 2nd player, the optimal control problem for the 1st player

\[
U(t, x, y, z) = \min_{\xi(\cdot) \in A} \mathbb{E} \left\{ \int_t^T \left( Z(s) + \lambda(\xi(s) + \psi(s, X(s), Y(s), Z(s))) \right) \cdot \xi(s) \, ds \right\} \quad \begin{cases}
X(t) = x \\
Y(t) = y \\
Z(t) = z
\end{cases}
\]

where

\[
\begin{align*}
\mathrm{d}Z(t) &= \left( \gamma \xi(t) + \gamma \psi(t, X(t), Y(t), Z(t)) \right) \mathrm{d}t + \sigma \mathrm{d}W(t) \\
\mathrm{d}X(t) &= \xi(t) \mathrm{d}t \\
\mathrm{d}Y(t) &= \psi(t, X(t), Y(t), Z(t)) \mathrm{d}t
\end{align*}
\]
The Stochastic Optimal Control Problem

The HJB equation for player 1

\[ U_t + \min \{ \lambda \xi^2 + \lambda \psi(t, x, y, z)\xi + z\xi + \xi U_x + \psi(t, x, y, z)U_y \\ + (\gamma \xi + \gamma \psi(t, x, y, z))U_z + \frac{1}{2} \sigma^2 U_{zz} \mid \xi \leq 0 \} = 0 \]

\[ -U_t = \psi(t, x, y, z)(U_y + \gamma U_z) + \frac{1}{2} \sigma^2 U_{zz} - \frac{1}{4 \lambda} [(z + \lambda \psi(t, x, y, z) + U_x + \gamma U_z)_+]^2 \]

for \( t \in [0, T] \), \( x \in [0, x_0] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}_+ \), and where

\[ \phi(t, x, y, z) = -\frac{1}{2\lambda} (z + \lambda \psi(t, x, y, z) + U_x + \gamma U_z)_+ \]
The Stochastic Optimal Control Problem

Given the CL strategy $\phi(\cdots)$ of the 1st player, the optimal control problem for the 2nd player

$$V(t, x, y, z) = \min_{\eta(\cdot) \in A} \mathbb{E} \left[ \int_t^T (Z(s) + \lambda(\phi(s, X(s), Y(s), Z(s)) + \eta(s))) \cdot \eta(s) \, ds \right]$$

where

$$\begin{align*}
    dZ(t) &= (\gamma \phi(t, X(t), Y(t), Z(t)) + \gamma \eta(t)) \, dt + \sigma \, dW(t) \\
    dX(t) &= \phi(t, X(t), Y(t), Z(t)) \, dt \\
    dY(t) &= \eta(t) \, dt
\end{align*}$$
The Stochastic Optimal Control Problem

The HJB equation for player 2

\[
V_t - \frac{1}{4\lambda}(z + \lambda\phi(t, x, y, z) + V_y + \gamma V_z)^2 \\
+ \phi(t, x, y, z)(V_x + \gamma V_z) + \frac{1}{2}\sigma^2 V_{zz} = 0
\]

for \( t \in [0, T] \), \( x \in [0, x_0] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}_+ \), and where

\[
\psi(t, x, y, z) = -\frac{1}{2\lambda}(z + \lambda\phi(t, x, y, z) + V_y + \gamma V_z)
\]
Clean up the entanglement of $\phi(\cdots)$ and $\psi(\cdots)$, we get

$$
\phi(t, x, y, z) = -\frac{1}{3\lambda} (z + 2U_x - V_y + 2\gamma U_z - \gamma V_z) +
\quad = -\frac{1}{3\lambda} (\delta(t, x, y, z)) +
$$

$$
\psi(t, x, y, z) = \begin{cases} 
-\frac{1}{3\lambda}(z - U_x + 2V_y - \gamma U_z + 2\gamma V_z) & \text{if } \delta(t, x, y, z) > 0 \\
-\frac{1}{2\lambda}(z + V_y + \gamma V_z) & \text{if } \delta(t, x, y, z) \leq 0
\end{cases}
$$

where $\delta(t, x, y, z) := z + 2U_x - V_y + 2\gamma U_z - \gamma V_z$. 
Define \( \delta(t, x, y, z) := z + 2U_x - V_y + 2\gamma U_z - \gamma V_z \)
and \( \delta^*(t, x, y, z) := z + 2V_y - U_x + 2\gamma V_z - \gamma U_z \)

where \( \delta(t, x, y, z) > 0, \)

\[
\begin{align*}
-U_t &= -\frac{1}{3\lambda} \delta^*(t, x, y, z)(U_y + \gamma U_z) \\
&+ \frac{1}{9\lambda} (\delta(t, x, y, z))^2 + \frac{1}{2} \sigma^2 U_{zz} \\
-V_t &= -\frac{1}{3\lambda} \delta(t, x, y, z)(V_x + \gamma V_z) \\
&+ \frac{1}{9\lambda} (\delta^*(t, x, y, z))^2 + \frac{1}{2} \sigma^2 V_{zz}
\end{align*}
\]

where \( \delta(t, x, y, z) \leq 0, \)

\[
\begin{align*}
-U_t &= -\frac{1}{2\lambda} (z + V_y + \gamma V_z)(U_y + \gamma U_z) + \frac{1}{2} \sigma^2 U_{zz} \\
-V_t &= -\frac{1}{4\lambda} (z + V_y + \gamma V_z)^2 + \frac{1}{2} \sigma^2 V_{zz}
\end{align*}
\]
In order to obtain stable numerical solutions, let us induce some viscosity condiments to the numerical scheme when solving the PDE system.

For the higher-order partials, instead of $\sigma^2 U_{zz}$ consider $\sigma^2 U_{zz} + \frac{1}{2} \epsilon \sigma^2 U_{xx} + \epsilon \sigma^2 U_{yy}$

and let $\epsilon \to 0$

Key verification: the numerical solution obtained is not sensitive at all to the choice of $\epsilon$
A trial example: the after-story of the liquidity trading game

The HJB equation for the 2nd player during the sequel period $[T, \bar{T}]$

\[
V_t + \min \left\{ \lambda \eta^2 + (z + V_y + \gamma V_z)\eta \mid \eta \leq 0 \right\} + \frac{1}{2} \epsilon \sigma^2 V_{yy} + \frac{1}{2} \sigma^2 V_{zz} = 0
\]

\[
-V_t = -\frac{1}{4\lambda} \left[ (z + V_y + \gamma V_z)_+ \right]^2 + \frac{1}{2} \sigma^2 V_{zz} + \frac{1}{2} \epsilon \sigma^2 V_{yy}
\]
Numerical Analysis of the NE

A trial example: the after-story of the liquidity trading game The HJB equation for the 2nd player during the sequel period $[T, \bar{T}]$

$$V_t + \min \left\{ \lambda \eta^2 + (z + V_y + \gamma V_z)\eta \mid \eta \leq 0 \right\}$$

$$+ \frac{1}{2} \epsilon \sigma^2 V_{yy} + \frac{1}{2} \sigma^2 V_{zz} = 0$$

$$-V_t = -\frac{1}{4\lambda} \left[ (z + V_y + \gamma V_z)_+ \right]^2 + \frac{1}{2} \sigma^2 V_{zz} + \frac{1}{2} \epsilon \sigma^2 V_{yy}$$
Numerical Analysis of the NE

A trial example: the after-story of the liquidity trading game. The HJB equation for the 2nd player during the sequel period $[T, \bar{T}]$

\[ V_t + \min \left\{ \lambda \eta^2 + (z + V_y + \gamma V_z)\eta \mid \eta \leq 0 \right\} \]

\[ + \frac{1}{2} \epsilon \sigma^2 V_{yy} + \frac{1}{2} \sigma^2 V_{zz} = 0 \]

\[-V_t = -\frac{1}{4\lambda} \left[ (z + V_y + \gamma V_z)_+ \right]^2 + \frac{1}{2} \sigma^2 V_{zz} + \frac{1}{2} \epsilon \sigma^2 V_{yy} \]
**Figure:** The numerical solution almost does not depend on the choice of coefficient for the artificial viscosity term. Here, $\epsilon = 2.5 \times 10^{-3}$
Figure: The numerical solution almost does not depend on the choice of coefficient for the artificial viscosity term. Here, $\epsilon = 1.0 \times 10^{-4}$
Figure: The numerical solution almost does not depend on the choice of coefficient for the artificial viscosity term.
Figure: The numerical solution almost does not depend on the choice of coefficient for the artificial viscosity term.
Volatility Does Enter the Picture and Make A Difference

Figure: The terminal value function for the predator under different volatility levels
Volatility Does Enter the Picture and Make A Difference

Figure: The terminal value function for the predator under different volatility levels
Summary and Ongoing Work

- Strategic interplay is an important source for Market Liquidity behaviors
- Adopt a reasonable market impact model, and think in *volume time*
- Closed-Loop strategies guarantee subgame perfectness, and usher volatility into the picture
- Numerical analysis of the NE of such a liquidity trading game
References


THANK YOU