Simulation of an SPDE Model for a Credit Basket

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Joint work with N. Bush, B. Hambly, H. Haworth

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Outline

- Introduction, structural models of credit
- Example: Credit default swap spreads
- A general multi-factor model
- Large basket limit and CDO pricing
- Numerical simulation of the SPDE
- Calibration and pricing examples
- Improvements, extensions
As in Merton (1974), and Black and Cox (1976),

- \( A_t \) the company’s asset value, governed by

\[
\frac{dA_t}{A_t} = \mu \, dt + \sigma \, dW_t
\]

- \( \mu \) the mean return, \( \sigma \) the volatility, \( W \) a standard Brownian motion

- denoting the default threshold barrier by \( b \), say constant, define the distance to default, \( X_t \), as

\[
X_t = \frac{1}{\sigma} \left( \log A_t - \log b \right).
\]

- Merton: company defaults if \( X_T < 0 \)

- Black/Cox: default time \( \tau \) is given by the first time \( X_t \) hits 0
By first exit time theory, the probability of survival to $T$ is

$$Q(T \leq \tau \mid X_t) = Q(X_s \geq 0, t \leq s \leq T \mid X_t)$$

$$= H(X_t, T - t),$$

where

$$H(x, s) = \phi \left( \frac{x + ms}{\sqrt{s}} \right) - e^{-2mx} \phi \left( \frac{-x + ms}{\sqrt{s}} \right).$$

$\Phi$ the standard Gaussian CDF, $m$ the risk-neutral drift of $X_s$. 

\[\]
Consider firm values, for $i = 1, \ldots, N$ (risk-neutral measure),

$$dA^i_t = (r_f - q_i) A^i_t \, dt + \sigma_i A^i_t \, dW_i(t)$$

where

- $r_f$ risk-free rate
- $q_i$ dividend yields
- $\sigma_i$ volatilities
- $W_i(t)$ Brownian motions and
- $\text{cov}(W_i(t), W_j(t)) = \rho_{ij} t$. 
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- \( \text{cov}(W_i(t), W_j(t)) = \rho_{ij} t \).

We assume that each company has an exponential default barrier,

\[
b_i(t) = K_i e^{-\gamma_i (T-t)}
\]

for constants \( K_i, \gamma_i \). \( T \) represents the maturity of the product.
Setting

\[ X_t^i = \ln \left( \frac{A_t^i}{A_0^i} e^{-\gamma_i t} \right) = \alpha_i t + \sigma_i W_i(t) \]

with \( \alpha_i = r_f - q_i - \gamma_i - \frac{1}{2} \sigma_i^2 \), leads to a Brownian motion with drift and constant barrier \( B_i = \ln \left( \frac{b_i(0)}{A_0^i} \right) \leq 0 \), \( X_0^i = 0 \).
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Defining the running minimum

\[ \overline{X}_t^i = \min_{0 \leq s \leq t} X_s^i, \]

and default time, \( \tau_i \), as the first hitting time of the default barrier,

\[ \tau^i = \inf \{ t : X_t^i = B_i \}, \]

the survival probability is then

\[ \mathbb{Q}(\tau^i > s) = \mathbb{Q}(\overline{X}_s^i \geq B_i). \]
For two firms, this problem is ‘analytically’ tractable, e. g.

\[ Q(t) = \mathbb{Q}(X^1_t \geq B_1, X^2_t \geq B_2) \]

\[ = \frac{2}{\beta t} e^{a_1 B_1 + a_2 B_2 + b t} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin \left( \frac{n \pi \theta_0}{\beta} \right) \int_0^\beta \sin \left( \frac{n \pi \theta}{\beta} \right) g_n(\theta) \, d\theta \]

where

\[ g_n(\theta) = \int_0^\infty r e^{-r^2/2t} e^{A(\theta)r} I_{\frac{n \pi}{\beta}} \left( \frac{r r_0}{t} \right) \, dr \]

\[ a_1 = \frac{\alpha_1 \sigma_2 - \rho \alpha_2 \sigma_1}{(1 - \rho^2) \sigma_1^2 \sigma_2}, \quad a_2 = \frac{\alpha_2 \sigma_1 - \rho \alpha_1 \sigma_2}{(1 - \rho^2) \sigma_1 \sigma_2^2} \]

\[ b = -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2} \sigma_1^2 a_1^2 + \rho \sigma_1 \sigma_2 a_1 a_2 + \frac{1}{2} \sigma_2^2 a_2^2 \]

\[ \tan \beta = -\frac{\sqrt{1 - \rho^2}}{\rho}, \quad \beta \in [0, \pi] \]

etc, and \( I_{\frac{n \pi}{\beta}} \left( \frac{r r_0}{t} \right) \) is a modified Bessel’s function.
The buyer of a $k^{th}$-to-default credit default swap (CDS) on a basket of $n$ companies pays a premium, the CDS spread, for the life of the CDS – until maturity or the $k^{th}$ default.

In the event of default by the $k^{th}$ underlying reference company, the buyer receives a default payment and the contract terminates.
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Equating discounted spread payment and discounted default payment,

$$\text{DSP} = cK \int_0^T e^{-rf_s} Q(\tau_k > s) \, ds$$

$$\text{DDP} = (1 - R)K \int_0^T e^{-rf_s} Q(s \leq \tau_k \leq s + ds)$$

gives the market $k^{th}$-to-default CDS spread, $c_k$, as

$$c_k = \frac{(1 - R) \left\{ 1 - e^{-rfT} Q(\tau_k > T) - \int_0^T r_f e^{-rf_s} Q(\tau_k > s) \, ds \right\}}{\int_0^T e^{-rf_s} Q(\tau_k > s) \, ds}.$$
Varying $T$

First-to-default CDS, varying $T$

Second-to-default CDS, varying $T$

\[
\sigma_1 = \sigma_2 = 0.2, K_1 = 100, r_f = 0.05, q_1 = q_2 = 0, \\
\gamma_1 = \gamma_2 = 0.03, \text{ initial distance-to-default} = 2, R = 0.5
\]
First-to-default CDS, varying $\sigma$

Second-to-default CDS, varying $\sigma$

\[ K_1 = 100, r_f = 0.05, q_1 = q_2 = 0, R = 0.5 \]
\[ \gamma_1 = \gamma_2 = 0.03, \text{ initial distance to default} = 2, T = 5 \]
Consider a portfolio of $N$ different companies.

Denoting the companies’ asset values at time $t$ by $A^i_t$, we assume that under the risk neutral measure they follow a diffusion process given by

$$dA^i_t = \mu(t, A^i_t) \, dt + \sigma(t, A^i_t) \, dW^i_t + \sum_{j=1}^{m} \sigma_{ij}(t, A^i_t) \, dM^j_t,$$

$W^i_t$ and $M^j_t$ are Brownian motions satisfying

$$d\langle W^i_t, M^j_t \rangle = 0 \quad \forall i, j$$

and

$$d\langle W^i_t, W^j_t \rangle = d\langle M^i_t, M^j_t \rangle = \delta_{ij} \, dt.$$
Solve Kolmogorov equation numerically?

Isotropic grid for $N = 2$ directions (firms):
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Isotropic grid for $N = 2$ directions (firms):
Numerical PDE solution

Solve Kolmogorov equation numerically?

Isotropic grid for $N = 2$ directions (firms):

`Curse of dimensionality`:

- After $n$ refinements, unknowns $\sim h^{-N} \sim 2^{nN}$
- Accuracy $\epsilon$ has complexity $C(\epsilon) \sim \epsilon^{-N/2}$ (for an order 2 method)

$\rightarrow$ exponential growth of required computational resources
Example, 30 firms: If each asset is represented by only two states, the total number of variables is already

\[ 2^{30} = 1 \, 073 \, 741 \, 824. \]
What is high dimensional?

Example, 30 firms: If each asset is represented by only two states, the total number of variables is already

\[ 2^{30} = 1\,073\,741\,824. \]

If we choose a reasonable number of points in each direction, say \(32 = 2^5\), the same total number is already obtained for

\[ \text{dim} = 6. \]
Sparse grids

Full grid, $h^{-N}$ points

Sparse, $h^{-1} |\log h|^{N-1}$

But: require too much smoothness and have ‘large constants’.
## Correlation data (typical)

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<th>Framework</th>
<th>Limit</th>
<th>Results</th>
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Spectral gap after $\lambda_1$.

Exponential decay from $\lambda_2$.

Asymptotic expansion? Not enough regularity.
It is irrelevant *which* of the firms default. Assume interchangeable.

- Let $\nu_{N,t}$ the empirical measure for the entire portfolio,

$$
\nu_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \delta_{A_t^i}
$$

- Let $\phi \in C_c^\infty(\mathbb{R})$ and for measure $\nu_t$ write

$$
\langle \phi, \nu_t \rangle = \int \phi(x) \nu_t(dx).
$$

- Define a family of processes $F_{t}^{N,\phi}$ by

$$
F_{t}^{N,\phi} = \langle \phi, \nu_{N,t} \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(A_t^i).
$$
Applying Ito's lemma to $F_t^{N,\phi}$ we have

$$
F_t^{N,\phi} = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \phi'(A_s^i) \left[ \mu(s, A_s^i) \, dt + \sigma(s, A_s^i) \, dW_s^i + \sum_{j=1}^{m} \sigma_{ij}(s, A_s^i) \, dM_s^j \right] 
+ \frac{1}{N} \sum_{i=1}^{N} \int_0^t \frac{1}{2} \phi''(A_s^i) \left( \sum_{j=1}^{m} \sigma_{ij}^2 + \sigma^2 \right) \, ds.
$$

We now assume that $\sigma_{ij} = \sigma_j$ for all $i = 1, 2 \ldots$, then

$$
F_t^{N,\phi} = F_0^{N,\phi} + \int_0^t \left< A\phi, \nu_{N,s} \right> \, ds + \int_0^t \sum_{j=1}^{m} \left< \sigma_j \phi', \nu_{N,s} \right> \, dM_s^j + 
+ \int_0^t \frac{1}{N} \sum_{i=1}^{N} \phi'(A_s^i) \sigma(s, A_s^i) \, dW_s^i.
$$

$$
A = \mu \frac{\partial}{\partial x} + \frac{1}{2} \bar{\sigma}^2 \frac{\partial^2}{\partial x^2}, 
\bar{\sigma}^2 = \sum_{j=1}^{m} \sigma_j^2 + \sigma^2,
$$
The idiosyncratic component becomes deterministic in the infinite dimensional limit, see also [Kurtz & Xiong, 1999].

As the market factors affect each company to the same degree, we can write

\[ F_{t}^{N,\phi} \to F_{t}^{\phi} = \langle \phi, \nu_{t} \rangle \quad \text{as} \quad N \to \infty, \]

with

\[ dF_{t}^{\phi} = \langle A\phi, \nu_{t} \rangle \, dt + \sum_{j=1}^{m} \langle \sigma_{j}\phi', \nu_{t} \rangle \, dM_{j}^{t}. \]

Alternatively, we can write this in integrated form as

\[ \langle \phi, \nu_{t} \rangle = \langle \phi, \nu_{0} \rangle + \int_{0}^{t} \langle A\phi, \nu_{s} \rangle \, ds + \sum_{j=1}^{m} \int_{0}^{t} \langle \sigma_{j}\phi', \nu_{s} \rangle \, dM_{j}^{s}. \]
Assume the measure $\nu_t$ to be absolutely continuous with respect to the Lebesgue measure, to write $\nu_t(dx) = v(t, x)dx$.

In differential form,

$$dv = -\mu \frac{\partial v}{\partial x} dt + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial x^2} dt - \sum_{j=1}^{m} \frac{\partial}{\partial x} (\sigma_j(t, x)v) dM^j_t.$$ 

This is a stochastic PDE that describes the evolution of an infinite portfolio of assets whose dynamics were given before.

For solubility see [Krylov, 1994].

Can now use this to approximate the loss distribution for a portfolio of fixed size $N$. 

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Assume asset processes are correlated via a single factor.

Specifically, let the SDEs for the asset processes be given by

\[
\frac{dA_t^i}{A_t^i} = r \, dt + \sqrt{1 - \rho \sigma} \, dW_t^i + \sqrt{\rho \sigma} \, dM_t^i, \quad A_0^i = a_i,
\]

where \( \rho \in [0, 1) \), \( d\langle W_t^i, M_t \rangle = 0 \).

The *distance-to-default*

\[
X_t^i = \frac{1}{\sigma} \left( \log A_t^i - \log B_t^i \right)
\]

evolves according to

\[
dX_t^i = \mu dt + \sqrt{1 - \rho} \, dW_t^i + \sqrt{\rho} \, dM_t, \quad X_0^i = x_i
\]

with \( \mu = \frac{1}{\sigma} \left( r - \frac{1}{2} \sigma^2 \right) \) and \( x^i = \frac{1}{\sigma} \left( \log a^i - \log B_0^i \right) \).
Portfolio loss

- Constant coefficient SPDE for $v$.
- The portfolio loss variable

$$L_t^N = (1 - R) \frac{1}{N} \sum_{i=1}^{N} 1_{\{\tau_i \leq t\}}$$

measures the losses at time $t$.

- Assume that a fraction $R$ of losses, $0 \leq R \leq 1$, the recovery rate, is recovered after default.

- Absorbing barrier condition:

$$v(t, 0) = 0 \quad \text{for } t \geq 0.$$ 

- The loss distribution is then

$$L_t = (1 - R) \left( 1 - \int_0^\infty v(t, x) \, dx \right).$$
Matching the initial conditions for both measures, we need

\[ v(0, x) = \frac{1}{N} \sum_{i=1}^{N} \delta_{A_0^i}. \]

Given this initial condition,

\[ L_0 = N \left( 1 - \frac{1}{N} \sum_{i=1}^{N} \int_{B_t} \delta_{A_0^i} \right) = 0, \]

Also,

\[ 0 \leq L_t \leq 1, \quad \text{for } t \geq 0 \]
\[ \mathbb{Q}(L_s \geq K) \leq \mathbb{Q}(L_t \geq K), \quad \text{for } s \leq t, \]

which ensure that there is no arbitrage in the loss distribution.
CDOs slice up the losses into tranches, defined by their attachment and detachment points.

Typical numbers are 0-3%, 3%-6%, 6%-9%, 9%-12%, 12%-22%, 22%-100%.

For an attachment point $a$ and detachment point $d > a$, the outstanding tranche notional

$$X_t = \max(d - L_t, 0) - \max(a - L_t, 0)$$

and tranche loss

$$Y_t = (d - a) - X_t = \max(L_t - a, 0) - \max(L_t - d, 0)$$

determine the spread and default payments for that tranche.

Spreads are quoted as an annual payment, as a ratio of the notional, but assumed to be paid quarterly.

\[\text{there is a variation for the equity tranche, but we do not go into details.}\]
Assume the notional to be 1, $c$ the spread payment;

$n$ the maximum number of payments up to expiry $T$;

$T_i$ the payment dates for $1 \leq i \leq n$, $\delta = 0.25$ the interval between payments.

The expected sum of discounted fee payments, referred to as the fee leg, is given by

$$cV^{fee} = c \sum_{i=1}^{n} \delta e^{-rT_i} \mathbb{E}^Q[X_{T_i}],$$

The protection leg is

$$V^{prot} = \sum_{i=1}^{n} e^{-rT_i} \mathbb{E}^Q[X_{T_{i-1}} - X_{T_i}].$$

The fair spread payment is then given by $s = \frac{V^{prot}}{V^{fee}}$. 
Recall the SPDE

\[
dv = -\frac{1}{\sigma} \left( r - \frac{1}{2} \sigma^2 \right) v_x \, dt + \frac{1}{2} v_{xx} \, dt - \sqrt{\rho} v_x \, dM_t, \quad t > 0, \ x > 0
\]

\[
v(0, x) = v_0(x), \quad x > 0
\]

\[
v(t, 0) = 0 \quad t \geq 0
\]

- Monte Carlo on top of a PDE solver computationally costly.
- For each simulated path of the market factor, a PDE needs to be solved, e.g. by a finite difference method.
- From the PDE solution, quantities like tranche losses can be computed,
- then averaged over the simulated paths.
First make the assumption that defaults are only observed at a discrete set of times,

taken quarterly to coincide with the payment dates,

i.e. if a firm’s value is below the default barrier on one of the observation dates $T_i$, removed from the basket.

We therefore solve the modified SPDE problem

$$dv = -\frac{1}{\sigma} \left( r - \frac{1}{2} \sigma^2 \right) v_x \, dt + \frac{1}{2} v_{xx} \, dt - \sqrt{\rho} v_x \, dM_t, \quad t \in (T_k, T_{k+1}),$$

$$v(0, x) = v_0(x),$$

$$v(T_k, x) = 0 \quad \forall x \leq 0, \quad 0 < k \leq n$$
The boundary condition is not active in intervals \((T_k, T_{k+1})\).

The Brownian driver only introduces a random shift.

The solution can therefore be written as

\[
v(t, x) = \begin{cases} 
0 & x \leq 0 \land t \in \{T_k, 1 \leq i \leq n\} \\
v^{(i)}(t - T_k, x - \sqrt{\rho}(M_t - M_{T_k})) & \text{else if } t \in (T_k, T_{k+1}], 0 \leq k < n
\end{cases}
\]

where \(v^{(k)}\) is the solution to the (deterministic) problem

\[
v^{(k)} = \frac{1}{2}(1 - \rho)v_{xx}^{(k)} - \frac{1}{\sigma} \left( r - \frac{1}{2}\sigma^2 \right) v_x^{(k)}, \quad t \in (0, \tau) = (0, T_{k+1} - T_k)
\]

\[
v^{(k)}(0, x) = v(T_k, x)
\]

assuming payment dates are equally spaced, \(\tau = T_{k+1} - T_k\).
This suggests the following inductive strategy for $k = 0, \ldots, n - 1$:

1. Start with $v^{(0)}(0, x) = v_0(x)$.
2. Solve the PDE numerically in the interval $(0, T_1)$, to obtain $v^{(0)}(T_1, x)$.
3. Simulate $M_{T_1}$, evaluate $v(T_1, x)$.
4. For $k > 0$, having computed $v(T_k, x)$ in the previous step, use this as initial condition for $v^{(k)}$, and repeat until $k = n$. 
The initial distribution is assumed localised.

Approximate the distribution by one with support $[x_{min}, x_{max}]$ with $x_{min} < 0$ and $x_{max} > 0$.

Can ensure that the expected error of this approximation is much smaller than the standard error of the Monte Carlo estimates.

Grid $x_0 = x_{min}, x_1 = x_{min} + \Delta x, \ldots, x_{min} + j\Delta x, \ldots, x_J = x_{min} + J\Delta x = x_{max}$, where $\Delta x = (x_{max} - x_{min})/J$,

timesteps $t_0 = 0, t_1 = \Delta t, \ldots, t_I = I\Delta t = \tau$, where $\Delta t = \tau/I$.

Define an approximation $v^i_j$ to $v(t_i, x_j)$ as solution to a FD/FE scheme with $\theta$ timestepping.
Standard central differencing is second order accurate in $\Delta x$.

The backward Euler scheme $\theta = 1$ is of first order accurate (in $\Delta t$) and strongly stable.

The Crank-Nicolson scheme $\theta = \frac{1}{2}$ is of second order accurate, and is unconditionally stable in the $l_2$-norm.

We deal with initial conditions of the form

$$v(0, x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x^i),$$

Crank-Nicolson timestepping gives spurious oscillations for Dirac initial conditions, and

reduces the convergence order for discontinuous initial conditions.
Rannacher proposed the following modification:

- Replace first Crank-Nicolson steps with backward Euler steps.
- Need to balance between accuracy and stability.
- Analysis by Giles and Carter of the heat equation suggests to replace the first two Crank-Nicolson steps by four backward Euler steps of half the stepsize.
- We do this at $t = 0$, and also at $t = T_k$ where the interface conditions introduce discontinuities at $x = 0$.
- This restores second order convergence in time.
The sum of $\delta$-distributions needs approximation on the grid.

Could collect the firms into symmetric intervals of width $\Delta x$ around grid points,

\[ v_j^0 = \frac{1}{N\Delta x} \int_{x_j-\Delta x/2}^{x_j+\Delta x/2} \delta(x^i - x) \, dx, \]

however, this reduces the overall order of the finite difference scheme to 1:

The approximation cannot distinguish between initial positions $x^i$ in an interval of length $\Delta x$. 
To achieve higher (i.e. second) order, the $\delta$'s are split between adjacent grid points. The correct weighting for a single firm with distance-to-default $x^i$ in the interval $[x_j, x_{j+1})$, is

$$v^0_k = \begin{cases} 
\Delta x^{-2} (x_{j+1} - x^i) & k = j \\
\Delta x^{-2} (x^i - x_j) & k = j + 1 \\
0 & \text{else}
\end{cases}$$

This can be written more elegantly as $L_2$-projection onto the basis of ‘hat functions’ $\langle \Phi_k \rangle_{0 \leq k \leq N}$ where

$$\Phi_k(x) = \frac{1}{\Delta x} \min (\max(x - x_k + \Delta x), \max(x_k - x, 0)),$$

then

$$v^0_k = \langle \Phi_k, v_0 \rangle = \int_{x_{\min}}^{x_{\max}} \Phi_k(x)v_0(x) \, dx.$$

Note that $\Delta x \sum_{k=0}^{N} v^0_k = 1.$
At $t = T_k$, we have to evaluate the grid function at shifted arguments that do not normally coincide with the grid.

- First, define a **piecewise linear reconstruction** from the approximation $v^I_k$ obtained in the last step over the previous interval $[T_{k-1}, T_k]$, as

$$v_{\Delta x}(T_k, x) = \sum_{j=0}^{N} \Phi_j(x - \Delta M) v^I_j.$$  

- Then, approximate the shift by setting

$$v^0_j = \int_{\max(x_j - \Delta x/2, 0)}^{x_j + \Delta x/2} v_{\Delta x}(T_k, x) \, dx,$$

with $\Delta M = M_{T_k} - M_{T_{k-1}}$. 
Use this as initial condition for the next interval (the integral is understood to be 0 if the lower limit is larger than the upper limit).

This ensures that

\[
\Delta x \sum_{j=0}^{J} v_j^0 = \int_{0}^{x_{max}} v_h(T_k, x) \, dx,
\]

so the cumulative density of firms with firm values greater than 0 is preserved.

Also, the solution is smoothened at \( x = 0 \)
For a given realisation of the market factor, we can approximate the loss functional $L_{T_k}$ at time $T_k$ by

$$L_{T_k}^{\Delta x} = 1 - \int_0^{x_{max}} v_{\Delta x}(T_k, x) \, dx$$

Explicitly include the dependency $L_{T_k}^{\Delta x}(\Phi)$, where $\Phi_i$ are drawn independently from a standard normal distribution,

Then for $N_{sims}$ simulations with samples $\Phi^l = (\Phi_k^l)_{1 \leq k \leq n}$, $1 \leq l \leq N_{sims}$,

$$\mathbb{E}^Q[X_{T_k}] \approx \mathbb{E}^Q[\max(d - L_{T_k}^{\Delta x}(\Phi), 0) - \max(a - L_{T_k}^{\Delta x}(\Phi), 0)]$$

$$\approx \frac{1}{N_{sims}} \sum_{l=1}^{N_{sims}} (\max(d - L_{T_k}^{\Delta x}(\Phi^l), 0) - \max(a - L_{T_k}^{\Delta x}(\Phi^l), 0))$$
The simulation error has two components, the discretisation error and the variance of the Monte Carlo estimate.

For the following simulations we have used these data:

\[ T = 5, \ r = 0.027, \ \sigma = 0.24, \ \log R = 0.7, \ \rho = 0.13. \]

Initial positions for individual firms, calibrated to their individual CDS spreads,

were well within the range \( [x_{min}, x_{max}] = [-10, 20] \).
First consider the discretisation error in $\Delta t$ and $\Delta x$.

- Extrapolation-based estimator for the discretisation error of $L_{T_n}$ for increasing $J$ (left) and $I$ (right) for a single realisation of the path of the market factor.

- We clearly see second order convergence in $\Delta t$ and $\Delta x$. 
<table>
<thead>
<tr>
<th>s</th>
<th>[0, 3%]</th>
<th>[3%, 6%]</th>
<th>[6%, 9%]</th>
<th>[9%, 12%]</th>
<th>[12%, 22%]</th>
<th>[22%, 100%]</th>
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<td>1</td>
<td>6.2725e-03</td>
<td>2.0969e-02</td>
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<td>2.80520e-02</td>
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<td>2.94938e-02</td>
<td>9.982806e-02</td>
<td>7.799986e-01</td>
</tr>
</tbody>
</table>

Monte Carlo estimates for expected outstanding tranche notional for $N_{sims} = 64 \times 4^{s-1}$ Monte Carlo runs, $s = 1, \ldots, 10$. The finite difference parameters were fixed at $J = 256, I = 4$. 

C. Reisinger – p.40
Monte Carlo estimates with standard error bars
for expected losses in tranches [0, 3%], [6%, 9%], [12%, 22%]
for $N_{sims} = 16 \cdot 4^{k-1}$, $k = 1, \ldots, 10$, $J = 256$, $I = 4$. 
The fixed coupons, traded spreads and model spreads for the iTraxx Main Series 10 index.

<table>
<thead>
<tr>
<th>Maturity Date</th>
<th>Fixed Coupon (bp)</th>
<th>Traded Spread (bp)</th>
<th>Model Spread (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20/12/2011</td>
<td>30</td>
<td>21</td>
<td>19.6</td>
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<tr>
<td>20/12/2013</td>
<td>40</td>
<td>30</td>
<td>30.7</td>
</tr>
<tr>
<td>20/12/2016</td>
<td>50</td>
<td>41</td>
<td>41.0</td>
</tr>
</tbody>
</table>

February 22, 2007. Parameters used for the model spreads are \( r = 0.042, \sigma = 0.22, \ R = 0.4 \).

<table>
<thead>
<tr>
<th>Maturity Date</th>
<th>Fixed Coupon (bp)</th>
<th>Traded Spread (bp)</th>
<th>Model Spread (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20/12/2013</td>
<td>120</td>
<td>215</td>
<td>207</td>
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<tr>
<td>20/12/2015</td>
<td>125</td>
<td>195</td>
<td>195</td>
</tr>
<tr>
<td>20/12/2018</td>
<td>130</td>
<td>175</td>
<td>176</td>
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</table>

December 5, 2008. Parameters used for the model spreads are \( r = 0.033, \sigma = 0.136, \ R = 0.4 \).
Implied Correlation Skew for iTraxx Main Series 6 Tranches, Feb 22, 2007. The implied correlation for each tranche is the value of correlation that gives a model tranche spread equal to the market tranche spread.

Model parameters are $r = 0.042, \sigma = 0.22, R = 0.4$. 
Model tranche spreads (bp) for varying values of the correlation parameter. The equity tranches are quoted as an upfront assuming a 500bp running spread.

The model is calibrated to the iTraxx Main Series 10 index for Dec 5, 2008. Market levels shown are for this date; model parameters are $r = 0.033, \sigma = 0.136, R = 0.4$. 

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Market</th>
<th>$\rho = 0.3$</th>
<th>$\rho = 0.4$</th>
<th>$\rho = 0.5$</th>
<th>$\rho = 0.6$</th>
<th>$\rho = 0.7$</th>
<th>$\rho = 0.8$</th>
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</thead>
<tbody>
<tr>
<td>0%-3%</td>
<td>71.5 %</td>
<td>81.88 %</td>
<td>75.9 %</td>
<td>69.56 %</td>
<td>63.02 %</td>
<td>56.25 %</td>
<td>49.16 %</td>
</tr>
<tr>
<td>3%-6%</td>
<td>1576.3</td>
<td>2275.2</td>
<td>1978.5</td>
<td>1743.2</td>
<td>1546.8</td>
<td>1374.6</td>
<td>1222.8</td>
</tr>
<tr>
<td>6%-9%</td>
<td>811.5</td>
<td>1273.1</td>
<td>1168.2</td>
<td>1079.7</td>
<td>1001.4</td>
<td>931.3</td>
<td>864.6</td>
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<td>9%-12%</td>
<td>506.1</td>
<td>775.7</td>
<td>765.8</td>
<td>748.6</td>
<td>724.7</td>
<td>695.8</td>
<td>663.2</td>
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<tr>
<td>12%-22%</td>
<td>180.3</td>
<td>307.8</td>
<td>353.3</td>
<td>384.7</td>
<td>405.5</td>
<td>418.1</td>
<td>423.4</td>
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<tr>
<td>22%-100%</td>
<td>77.9</td>
<td>9.2</td>
<td>16.5</td>
<td>25</td>
<td>34.3</td>
<td>44.5</td>
<td>55.7</td>
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### Results

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Market</th>
<th>( \rho = 0.3 )</th>
<th>( \rho = 0.4 )</th>
<th>( \rho = 0.5 )</th>
<th>( \rho = 0.6 )</th>
<th>( \rho = 0.7 )</th>
<th>( \rho = 0.8 )</th>
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</thead>
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<tr>
<td>0%-3%</td>
<td>72.9%</td>
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<td>73.26%</td>
<td>66.93%</td>
<td>60%</td>
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<td>3%-6%</td>
<td>1473.2</td>
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<td>1985.7</td>
<td>1715.2</td>
<td>1493.4</td>
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<td>6%-9%</td>
<td>804.2</td>
<td>1344.2</td>
<td>1199</td>
<td>1085.2</td>
<td>988.2</td>
<td>900.7</td>
<td>820.9</td>
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<tr>
<td>9%-12%</td>
<td>512.4</td>
<td>855.4</td>
<td>808.4</td>
<td>765.3</td>
<td>725.3</td>
<td>684.8</td>
<td>643</td>
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<tr>
<td>12%-22%</td>
<td>182.6</td>
<td>375.4</td>
<td>401.7</td>
<td>417.6</td>
<td>425.6</td>
<td>427.4</td>
<td>423.1</td>
</tr>
<tr>
<td>22%-100%</td>
<td>75.8</td>
<td>14</td>
<td>22</td>
<td>30.6</td>
<td>39.6</td>
<td>49.3</td>
<td>59.7</td>
</tr>
</tbody>
</table>

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Extensions

- Dynamic properties, forward starting CDOs, options on tranches,...
- Simulation of SPDEs driven by jumps, with Karolina Bujok.
- Calibration, including jumps.
- Analytic work by Hambly/Jin on jumps, contagion, granularity.
- Continuous defaults, with Mike Giles, via Multi-Level Monte Carlo.
- Convergence analysis through mean-square stability.
- Importance sampling for senior tranches, with Tom Dean.
Using large deviations theory, see also [Glasserman et al, 1999, 2007], preliminary observations:

**Probability of more than 3% defaults occurring for $1000 \times 2^n$ Monte Carlo paths**
Using large deviations theory, see also [Glasserman et al, 1999, 2007], preliminary observations:

**Probability of more than 10% defaults occurring for $1000 \times 2^n$ Monte Carlo paths**
References


