On robust pricing and hedging and the resulting notions of weak arbitrage

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based on joint works with
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Outline

Principal Questions and Answers
   Financial Problem  (2 questions)
   Methodology      (2 answers)

Double barrier options
   Introduction and types of barriers
   Double no-touch example

Theoretical framework and arbitrages
   Pricing operators and arbitrages
   No arbitrage vs existence of a model
Robust techniques in quantitative finance

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Robust methods: principal ideas

Model risk:

- Any given model is unlikely to capture the reality.
- Strategies which are sensitive to model assumptions or changes in parameters are questionable.
- We look for strategies which are robust w.r.t. departures from the modelling assumptions.

Market input:

- We want to start by taking information from the market. E.g. prices of liquidly traded instruments should be treated as an input.
- We can then add modelling assumptions and try to see how these affect, for example, admissible prices and hedging techniques.
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Robust pricing and hedging: 2 questions

The general setting and challenge is as follows:

- Observe prices of some liquid instruments which admit no arbitrage. (→ interesting questions!)
- **Q1**: *(very) robust pricing*
  Given a new product, determine its feasible price, i.e. range of prices which do not introduce an arbitrage in the market.
- **Q2**: *(very) robust hedging*
  Furthermore, derive tight super-/sub- hedging strategies which always work.

E.g.: Put-Call parity, Up-and-in put
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Q1 and the Skorokhod Embedding Problem

Q1: *What is the range of no-arbitrage prices of an option $O_T$ given prices of European calls?*

- Suppose:
  - $(S_t)$ is a continuous martingale under $\mathbb{P} = \mathbb{Q}$,
  - we see market prices $C_T(K) = \mathbb{E}(S_T - K)^+$, $K \geq 0$.
- Equivalently $(S_t : t \leq T)$ is a UI martingale, $S_T \sim \mu$,
  $$\mu(dx) = C''(x)dx.$$
- Via Dubins-Schwarz $S_t = B_{\tau_t}$ is a time-changed Brownian motion. Say we have $O_T = O(S)_T = O(B)_{\tau_T}$.
- We are led then to investigate the bounds
  $$LB = \inf_{\tau} \mathbb{E}O(B)_{\tau}, \quad \text{and} \quad UB = \sup_{\tau} \mathbb{E}O(B)_{\tau},$$
  for all stopping times $\tau$: $B_\tau \sim \mu$ and $(B_{t\wedge \tau})$ a UI martingale, i.e. for all solutions to the Skorokhod Embedding problem.
- The bounds are tight: the process $S_t := B_{\tau_t \wedge \frac{t}{T-t}}$ defines an asset model which matches the market data.
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Q2 and pathwise inequalities

**Q2**: *if we see a price outside the bounds \((LB, UB)\) can we (and how) realise a risk-less profit?*

- Consider UB. The idea is to devise inequalities of the form

  \[ O(B)_t \leq N_t + F(B_t), \quad t \geq 0, \]

  with equality for some \(\tau^*\) with \(B_{\tau^*} \sim \mu\), and where \(N\) is a martingale (i.e. trading strategy), \(\mathbb{E}N_{\tau^*} = 0\).

- Then \(UB = \mathbb{E}F(S_T)\) and \(+ F(S_t)\) is a valid superhedge. It involves dynamic trading and a static position in calls \(F(S_T)\).

- Furthermore, we want \((N_{\tau_t})\) explicitly. We are naturally restricted to the family of martingales \(N_t = N(B_t, A_t)\), for some process \((A_t)\) related to the option \(O_t\), e.g. maximum and minimum processes for barrier options.
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Scope of applications

- **Answer to Q1 and pricing**: in practice $LB \ll UB$, the bounds are *too wide* to be of any use for pricing.
- **Answer to Q2 and hedging**: say an agent sells $O_T$ for price $p$. She then can set up our super-hedge for $UB$. At the expiry she holds

$$X = p - UB + F(S_T) + N(S_T, A^S_T) - O_T.$$  

We have $\mathbb{E}^Q X = 0$ and $X \geq p - UB$. The hedge might have a considerable variance but the loss is bounded below (for all $t \leq T$). The hedge is very robust as we make virtually no modelling assumptions and only use market input. This can be advantageous in presence of

- model uncertainty
- transaction costs
- illiquid markets.

Numerical simulations indicate that a risk averse agent prefers robust hedges to delta/vega hedging.
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References and current works

Previous works adapting the strategy:


As well as:

- *forward starting options* (D. Hobson and A. Neuberger, ...)
- *volatility derivatives* (B. Dupire, R. Lee, ...)
- *double barrier options* (A. Cox and J.O., arxiv: 0808.4012, 0901.0674 ...)
- *variance swaps* (M. Davis, J.O. and V. Raval)
Double barriers - introduction

We want to apply the above methodology to derivatives with digital payoff conditional on the stock price reaching/not reaching two levels.

Continuity of paths implies level crossings (i.e. payoffs) are not affected by time-changing.

An example is given by a double touch:

\[
1_{\sup_{t \leq T} S_t \geq b \text{ and } \inf_{t \leq T} S_t \leq b}.
\]

\[
\leadsto 1_{\sup_{u \leq \tau} B_u \geq b \text{ and } \inf_{u \leq \tau} B_u \leq b}.
\]

In general the option pays 1 on the event

\[
\left\{ \sup_{t \leq T} S_t \left( \begin{array}{c} \leq b \\ \geq \end{array} \right) \text{ (and) } \inf_{t \leq T} S_t \left( \begin{array}{c} \leq b \\ \geq \end{array} \right) \right\}
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In general the option pays 1 on the event

\[ \left\{ \sup_{t \leq T} S_t \begin{pmatrix} \leq \end{pmatrix} b \begin{pmatrix} \text{ and } \end{pmatrix} \inf_{t \leq T} S_t \begin{pmatrix} \geq \end{pmatrix} b \right\}. \]
There are 8 possible digital double barrier options. However using complements and symmetry, it suffices to consider 3 types:

- **double touch option** $\leadsto$ new solutions to the SEP.
- **double touch/no-touch option** $\leadsto$ new solutions to the SEP.
- **double no-touch option** $\leadsto$ maximised by Perkins’ construction and minimised by the tilted-Jacka (A. Cox) construction.
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- **double touch/no-touch option** $\rightsquigarrow$ new solutions to the SEP.
- **double no-touch option** $\rightsquigarrow$ maximised by Perkins’ construction and minimised by the tilted-Jacka (A. Cox) construction.
Double no-touch: Answer to Q1

- Write $\bar{B}_t = \sup_{s \leq t} B_s$, $\underline{B}_t = \inf_{s \leq t} B_s$:

$$\inf\{t : B_t \notin (\gamma-(\bar{B}_t), \gamma+(\underline{B}_t))\}$$

- Maximises:

$$\mathbb{P}(B_\tau \geq b \text{ and } \bar{B}_\tau \leq \bar{b})$$

- Perkins (1985)
Double no–touch: Answer to Q1

- Write $\overline{B}_t = \sup_{s \leq t} B_s$, $\underline{B}_t = \inf_{s \leq t} B_s$:

  \[ \inf\{t : B_t \notin (\gamma_-(\overline{B}_t), \gamma_+(-\overline{B}_t))\} \]

- Maximises:

  \[ \mathbb{P}(\overline{B}_\tau \geq b \text{ and } \overline{B}_\tau \leq \overline{b}) \]

- Perkins (1985)
Double no–touch: Answer to Q2

- Consider pathwise inequality:

\[
\mathbf{1}_{\{S_T \geq b, S_T \leq b\}} \leq \mathbf{1}_{S_T > b} \left(\frac{(b - S_T)^+}{K - b} + \frac{(S_T - K)^+}{K - b} - \frac{S_T - b}{K - b}\right) \mathbf{1}_{S_T \leq b}
\]

where \(b < S_0 < K\).

- When \(S_T > b\), we get:

\[
1 \leq \mathbf{1}_{S_T > b} + \frac{(S_T - K)^+}{K - b}
\]
Double no-touch: Answer to Q2

- Consider pathwise inequality:

\[ 1 \{ S_T \geq b, S_T \leq b \} \leq 1_{S_T > b} \frac{(b - S_T)^+}{K - b} + \frac{(S_T - K)^+}{K - b} - \frac{S_T - b}{K - b} 1_{S_T \leq b} \]

where \( b < S_0 < K \).

- When \( S_T \leq b \), we get:

\[ 0 \leq \frac{(K - S_T)^+}{K - b} 1_{\{S_T > b\}} \]
Double no-touch: Answer to Q2

- Consider pathwise inequality:

\[ 1_{\{S_T \geq b, S_T \leq b\}} \leq 1_{S_T > b} \left( \frac{(b - S_T)^+}{K - b} + \frac{(S_T - K)^+}{K - b} \right) - \frac{S_T - b}{K - b} 1_{S_T \leq b} \]

where \( b < S_0 < K \).

This is a model-free superhedging strategy for any \( b < K \).
Double no–touch: Answer to Q2

Consider pathwise inequality:

\[ 1_{\{S_T \geq b, \overline{S}_T \leq \overline{b}\}} \leq 1_{S_T > b} \left( \frac{(b - S_T)^+}{K - b} \right) - \frac{(S_T - K)^+}{K - b} \]

Digital call

\[ - \frac{S_T - b}{K - b} 1_{S_T \leq b} \]

Puts

\[ \text{Calls} \]

Forwards upon hitting \( b \)

\[ =: \overline{H}^{ll}(K) \]

This is a model–free superhedging strategy for any \( b < K \), assuming \((S_t)\) does not jump across the barrier \( b \).
**Double no-touch: Answer to Q2**

- Consider pathwise inequality:

\[
1_{\{S_T \geq b, S_T \leq \bar{b}\}} \leq 1_{S_T > b} - \underbrace{\frac{(b - S_T)^+}{K - b}}_{\text{Digital call}} \underbrace{\frac{(S_T - K)^+}{K - b}}_{\text{Calls}} + \underbrace{\frac{S_T - b}{K - b} 1_{S_T \leq b}}_{\text{Puts}} =: \overline{H}^{II}(K)
\]

We would like to show that it is a hedging strategy in some model. It turns out that the above construction is not always optimal — there are two more strategies \(\overline{H}^I, \overline{H}^{III}(K)\) we need to consider. Above we superheded \(1_{S_T > b}\) as in Brown, Hobson, Rogers (2001) and it’s good only for \(b < S_0 << \bar{b}\).
Double touch: superhedging

Write $\mathcal{P}$ for the pricing operator. No arbitrage should imply:

$$\mathcal{P}1_{\{S_T \geq b, S_T \leq \bar{b}\}} \leq \inf \left\{ \mathcal{P}H^I, \mathcal{P}H^{II}(K_2), \mathcal{P}H^{III}(K_3) \right\} =: UB \quad (\dagger)$$

where the infimum is taken over values of $K_2 > b$, $K_3 < \bar{b}$.

Theorem (”Meta-Theorem”)

No arbitrage iff $(\dagger)$ holds and for any given curve of call prices there exists a stock price process for which $(\dagger)$ is the price of the double no–touch option.
Double touch: superhedging

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**Theorem ("Meta-Theorem")**

No arbitrage *iff* $(\dagger)$ holds *and* for any given curve of call prices there exists a stock price process for which $(\dagger)$ is the price of the double no–touch option.
Principal Questions and Answers

Double barrier options

Theoretical framework and arbitrages

General setup

We assume \((S_t : t \leq T)\) takes values in some functional space \(\mathcal{P}\), and \((S_t)\) has zero cost of carry (e.g. interest rates are zero). The set of traded assets \(\mathcal{X}\) is given. On this set we have a pricing operator \(\mathcal{P}\) which acts linearly on \(\mathcal{X}\), \(\mathcal{P} : \text{Lin}(\mathcal{X}) \to \mathbb{R}\).

We say that there exists a \((\mathcal{P}, \mathcal{X})\)-market model if there is a model \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}, (S_t))\) with \(\mathcal{P}X = E^\mathbb{Q}X, X \in \mathcal{X}\).

We would like to have

\(\mathcal{P}\) admits no arbitrage on \(\mathcal{X}\) \iff there exists a market model

Then we want to consider \(\mathcal{X} \cup \{O_T\}\) for an exotic \(O_T : \mathcal{P} \to \mathbb{R}\) and say

\(\mathcal{P}\) admits no arbitrage on \(\mathcal{X} \cup \{O_T\}\) \iff \(LB \leq \mathcal{P}O_T \leq UB\)

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\[ \mathcal{P} \text{ admits no arbitrage on } \mathcal{X} \cup \{O_T\} \iff LB \leq \mathcal{P}O_T \leq UB \]

\[ \iff \text{there exists a } (\mathcal{P}, \mathcal{X} \cup \{O_T\})\text{-market model} \]
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We assume \((S_t : t \leq T)\) takes values in some functional space \(\mathcal{P}\), and \((S_t)\) has zero cost of carry (e.g. interest rates are zero). The set of traded assets \(\mathcal{X}\) is given. On this set we have a **pricing operator** \(\mathcal{P}\) which acts linearly on \(\mathcal{X}\), \(\mathcal{P} : \text{Lin}(\mathcal{X}) \to \mathbb{R}\).

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\(\mathcal{P}\) admits **no arbitrage** on \(\mathcal{X} \cup \{O_T\}\) \(\iff\) \(LB \leq \mathcal{P}O_T \leq UB\) \(\iff\) there exists a \((\mathcal{P}, \mathcal{X} \cup \{O_T\})\)-market model
Three notions of arbitrage

**Definition (Model–free arbitrage)**

We say that $\mathcal{P}$ admits a **model–free arbitrage** on $\mathcal{X}$ if there exists $X \in \text{Lin}(\mathcal{X})$ with $X \geq 0$ and $\mathcal{P}X < 0$. 
Three notions of arbitrage

**Definition (Model–free arbitrage)**

We say that $\mathcal{P}$ admits a model–free arbitrage on $\mathcal{X}$ if there exists $X \in \text{Lin}(\mathcal{X})$ with $X \geq 0$ and $\mathcal{P}X < 0$.

This coarsest notion is typically sufficient to derive no–arbitrage bounds but *not* sufficient to give existence of a market model. Consider $\mathcal{X} = \{(S_T - K)^+ : K \in K = \{K_1, \ldots, K_n\}\}$. No MFA implies interpolation of $C(K) := \mathcal{P}(S_T - K)^+$ is convex and non-increasing. We could have $C(K_{n-1}) = C(K_n) > 0$. But this leads to arbitrage strategies:

- if I have a model with $S_T \leq K_n$ a.s., I sell call with strike $K_n$,
- if I have a model with $\mathbb{P}(S_T > K_n) > 0$ I sell call with strike $K_n$ and buy call with strike $K_{n-1}$.
Three notions of arbitrage

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Definition (Weak arbitrage (Davis & Hobson 2007))

We say that $\mathcal{P}$ admits a weak arbitrage on $\mathcal{X}$ if for any model, there exists $X \in \text{Lin}(\mathcal{X})$ with $\mathcal{P}X \leq 0$ but $\mathbb{P}(X \geq 0) = 1$, $\mathbb{P}(X > 0) > 0$.

Definition (Weak free lunch with vanishing risk)

We say that $\mathcal{P}$ admits a weak free lunch with vanishing risk on $\mathcal{X}$ if there exists $X_n, Z \in \text{Lin}(\mathcal{X})$ such that $X_n \rightarrow X$ (pointwise on $\mathcal{P}$), $X_n \geq Z$, $X \geq 0$ and $\lim \mathcal{P}X_n < 0$. 
Three notions of arbitrage

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Call prices and no arbitrages

Proposition (Davis and Hobson (2007))

Let $\mathcal{X} = \{1, (S_T - K)^+ : K \in \mathbb{K}\}$ be finite. Then $\mathcal{P}$ admits no WA on $\mathcal{X}$ if and only if there exists a $(\mathcal{P}, \mathcal{X})$-market model.
Call prices and no arbitrages

Proposition

Let $\mathcal{X} = \{1, (S_T - K)^+ : K \geq 0\}$. Then $\mathcal{P}$ admits no WFLVR on $\mathcal{X}$ if and only if there exists a $(\mathcal{P}, \mathcal{X})$-market model, which happens if and only if

$$C(K) = \mathcal{P}(S_T - K)^+ \geq 0$$

is convex and non-increasing,

and $C(0) = S_0$, $C'_+(0) \geq -1$,

$$C(K) \to 0 \text{ as } K \to \infty.$$  \hspace{1cm} (1)

In comparison, $\mathcal{P}$ admits no model-free arbitrage on $\mathcal{X}$ if and only if (1) holds. In consequence, when (1) holds but (2) fails $\mathcal{P}$ admits no model-free arbitrage but a market model does not exist.
Call and digital call prices and no arbitrages

Proposition

Let \( \mathcal{X} = \{ 1, 1_{S_T > b}, 1_{S_T \geq \bar{b}}, (S_T - K)^+ : K \geq 0 \} \). Then \( \mathcal{P} \) admits no WFLVR on \( \mathcal{X} \) if and only if there exists a \( (\mathcal{P}, \mathcal{X}) \)-market model, which happens if and only if \( C(K) \) is as previously and

\[
\mathcal{P} 1_{S_T > b} = -C'(b+) \quad \text{and} \quad \mathcal{P} 1_{S_T \geq \bar{b}} = -C'(\bar{b}-).
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Let \( \mathcal{X} = \{1, 1_{S_T > b}, 1_{S_T \geq b}, (S_T - K)^+ : K \in \mathbb{K}\} \) be finite. Then \( \mathcal{P} \) admits no WA on \( \mathcal{X} \) if and only if there exists a \((\mathcal{P}, \mathcal{X})\)-market model.

In both cases no WFLVR or no WA are strictly stronger than no model–free arbitrage.
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Double barriers and no–arbitrage

Theorem

Let $\mathcal{P} = C([0, T])$. Suppose $\mathcal{P}$ admits no WFLVR on $\mathcal{X} = \{\text{forwards}\} \cup \{1, 1_{S_T > b}, 1_{S_T \geq \bar{b}}, (S_T - K)^+: K \geq 0\}$. Then the following are equivalent:

- $\mathcal{P}$ admits no WFLVR on $\mathcal{X} \cup \{1_{S_T \geq \bar{b}, S_T \leq b}\}$,
- there exists a $(\mathcal{P}, \mathcal{X} \cup \{1_{S_T \geq \bar{b}, S_T \leq b}\})$ market model,
- 

$$
\mathcal{P}(1_{S_T \geq \bar{b}, S_T \leq b}) \leq \inf \left\{ \mathcal{P}(\overline{H}^I), \mathcal{P}(\overline{H}^{II}(K_2)), \mathcal{P}(\overline{H}^{III}(K_3)) \right\},
$$

$$
\mathcal{P}(1_{S_T \geq \bar{b}, S_T \leq b}) \geq \sup \left\{ \mathcal{P}(H^I), \mathcal{P}(H^{II}(K_1, K_2)) \right\}.
$$

(and we specify the hedges & strike(s) which attain inf/sup).

All our main results for digital double barriers are of this type with WA replacing WLVR for the case of finite family of traded strikes.
Summary

- Given a set of traded assets we want to construct robust super- and sub- hedging strategies of an exotic option. Further, we want them to be optimal in the sense that there exists a model, matching the market input, in which they are the hedging strategies.

- We carry out this programme for all types of digital double barrier options when the set of traded assets includes calls, digital calls and forward transactions.

- We introduce a formalism for the model–free setup and define stronger notions of arbitrage (WFLVR and WA).

- There exists a market model (matching the input) iff appropriate no–arbitrage holds. Further, the same holds if we add a double barrier, and this is equivalent to its price being within the bounds we derive.
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