Basis risk with random parameters

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Outline

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Related literature

- Basis risk models:
  - Davis [3], Henderson [5], MM [8, 9, 10], Musiela and Zariphopoulou [11], Ankirchner, Imkeller and Reis [1]

- Exponential utility maximisation with random endowment:
  - Delbaen et al [4], Mania and Schweizer [7]

- Asymptotic analysis of utility-based prices for small numbers of claims:
  - Kramkov and Sîrbu [6]

- Partial information models:
  - Rogers [13], Björk, Davis and Landén [2]
Random parameter basis risk model

- $(\Omega, \mathcal{F}, P)$, $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$
- $(S, Y) = (S_t, Y_t)_{0 \leq t \leq T}$

$$dS_t = \sigma_t^S S_t (\lambda_t^S dt + dB_t^S), \quad dY_t = \sigma_t^Y Y_t (\lambda_t^Y dt + dB_t^Y)$$

where

$$B^Y = \rho_t B^S + \sqrt{1 - \rho_t^2} Z^S, \quad \rho_t \in [-1, 1]$$

- $\mathbb{F}$-adapted parameters
- $r = 0$
- $\sigma^S, \sigma^Y, \rho$ also $\mathbb{F}$-adapted
- Markovian model: $\nu_t \equiv \nu(t, S_t, Y_t)$, for $\nu \in \{\sigma^S, \sigma^Y, \lambda^S, \lambda^Y, \rho\}$
- Claim $C(Y_T) \geq 0$, bounded, $\mathcal{F}_T$-measurable
Exponential valuation and hedging

For $t \in [0, T]$, given $X_t = x$, portfolio wealth process is

$$X_u = x + \int_t^T \theta_u dS_u = x + \int_t^T \pi_u (\lambda^S_u du + dB^S_u), \quad t \leq u \leq T$$

where $\pi := \theta S$. Denote by $\Theta$ (or $\Pi$) the set of admissible $\theta$ (or $\pi$)

$$U(x) := \exp(-\alpha x), \quad x \in \mathbb{R}, \quad \alpha > 0$$

Given $(X_t, S_t, Y_t) = (x, s, y)$, primal value function is

$$u^C(t, x, s, y) := \sup_{\pi \in \Pi} E_{t,x,s,y}[U(X_T - C(Y_T))] \quad (1)$$

Indifference price $p(t, s, y)$ defined by

$$u^C(t, x + p(t, s, y), s, y) = u^0(t, x, s, y)$$

Denote optimal strategy for (1) by $\pi^C$. Optimal hedging strategy $\pi^{(H)}$ defined by

$$\pi^{(H)} := \pi^C - \pi^0$$
Entropy and admissibility

\[ \mathbb{P}_e := \{ Q \sim P | S \text{ is a local } (Q, \mathbb{F})\text{-martingale} \} \]

\[ \mathbb{P}_{e,f} := \{ Q \in \mathbb{P}_e | H(Q, P) < \infty \} \neq \emptyset \]

\[ H(Q, P) := E \left[ \frac{dQ}{dP} \log \frac{dQ}{dP} \right], \quad \text{(if finite, else } H(Q, P) := \infty) \]

For \( Q \in \mathbb{P}_{e,f} \), define \( P \)-martingale \( \Gamma^Q \) by

\[ \Gamma^Q_t := \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \mathcal{E}(-\lambda^S \cdot B^S - \psi \cdot Z^S), \quad 0 \leq t \leq T \]

Admissible strategies

\[ \Theta := \{ \theta | (\theta \cdot S) \text{ is a } (Q, \mathbb{F})\text{-martingale for all } Q \in \mathbb{P}_{e,f} \} \]
Duality

Dual value function

\[ \tilde{u}(t, \eta, s, y) := \inf_{Q \in \mathcal{P}_{e, f}} E_{t,s,y} \left[ \tilde{U} \left( \eta \frac{\Gamma^Q}{\Gamma_t^Q} \right) - \eta \frac{\Gamma^Q}{\Gamma_t^Q} C(Y_T) \right] \]

where, for exponential utility

\[ \tilde{U} = \frac{\eta}{\alpha} \left[ \log \left( \frac{\eta}{\alpha} \right) - 1 \right] \]

Hence

\[ \tilde{u}(t, \eta, s, y) = \tilde{U}(\eta) + \frac{\eta}{\alpha} H^C(t, s, y) \]

where

\[ H^C(t, s, y) := \inf_{Q \in \mathcal{P}_{e, f}} E_{t,s,y}^Q \left[ \log \left( \frac{\Gamma^Q}{\Gamma_t^Q} \right) - \alpha C(Y_T) \right] \]

Primal and dual value functions conjugate, so primal problem has representation

\[ u^C(t, x, s, y) = -\exp \left( -\alpha x - H^C(t, s, y) \right) \] (2)
Dual control problem

\[ B_t^{s,Q} := B_t^s + \int_0^t \lambda_s^s du, \quad Z_t^{s,Q} := Z_t^s + \int_0^t \psi_u du, \quad 0 \leq t \leq T \]

(\( \psi \equiv 0 \) corresponds to \( Q^M \)). Then, for \( Q \in \mathbb{P}_{e,f} \),

\[ E_{t,s,y}^Q \log \frac{\Gamma_T^Q}{\Gamma_t^Q} = E_{t,s,y}^Q \frac{1}{2} \int_t^T \left[ (\lambda_u^s)^2 + \psi_u^2 \right] du < \infty \quad (3) \]

Let \( \Psi \) denote set of \( \psi \) such that (3) is satisfied. Then \( H^C \) is value function of stochastic control problem

\[ H^C(t, s, y) := \inf_{\psi \in \Psi} E_{t,s,y}^Q \left[ \frac{1}{2} \int_t^T \left[ (\lambda_u^s)^2 + \psi_u^2 \right] du - \alpha C(Y_T) \right] \quad (4) \]

where, under \( Q \in \mathbb{P}_{e,f} \), state variables \( S, Y \) follow

\[
\begin{align*}
    dS_t &= \sigma_t^S S_t dB_t^{s,Q} \\
    dY_t &= \sigma_t^Y Y_t \left[ (\lambda_t^Y - \rho_t \lambda_t^S - \sqrt{1 - \rho_t^2 \psi_t}) dt + dB_t^{Y,Q} \right] \\
    B_t^{Y,Q} &= \rho_t B_t^{s,Q} + \sqrt{1 - \rho_t^2} Z_t^{s,Q}
\end{align*}
\]
Dual representation of indifference price

For $C \equiv 0$,

$$H^0(0, \cdot, \cdot) = H(Q^E, P)$$

Write $H^0(t, s, y) \equiv H^E(t, s, y)$, the “minimal entropy” value function. Then

$$u^0(t, x, s, y) = -\exp(-\alpha x - H^E(t, s, y))$$  \hspace{1cm} (5)

Indifference price definition plus (2) and (5) give

$$u^C(t, x, s, y) = u^0(t, x, s, y) \exp(\alpha p(t, s, y))$$

and indifference price has entropic representation

$$-\alpha p(t, s, y) = H^C(t, s, y) - H^E(t, s, y)$$
Optimal hedging strategy

**Theorem**

Optimal hedge: hold $\theta_t^{(H)}$ shares of $S_t$ at $t \in [0, T]$, where

$$
\theta_t^{(H)} = \left( \frac{\partial p}{\partial s}(t, S_t, Y_t) + \rho(t, S_t, Y_t) \frac{\sigma^Y(t, S_t, Y_t)}{\sigma^S(t, S_t, Y_t)} \frac{Y_t}{S_t} \frac{\partial p}{\partial y}(t, S_t, Y_t) \right)
$$

**Remark**

Additional term $p_s(t, S_t, Y_t)$ compared with earlier studies. Partial information model of Section 3 is of this form, and $p_s(t, S_t, Y_t)$ reflects additional risk induced by parameter uncertainty.
Proof.

Use HJB equation associated with primal the value function

\[
\frac{\partial u^C}{\partial t} + \max_{\pi} A_{\pi,S,Y} u^C = 0
\]

Compute optimal Markov control and use separable form (2) of value function, to obtain optimal strategy as \( \pi_t^C = \pi^C(t, S_t, Y_t) \), where

\[
\pi^C(t, s, y) = \frac{\lambda^S}{\sigma^S \alpha} - \frac{1}{\alpha} \left( s(H^E_s - \alpha p_s) + \rho \frac{\sigma^Y}{\sigma^S} y(H^E_y - \alpha p_y) \right)
\]

with similar form for \( C = 0 \). Apply definition of optimal hedging strategy:

\[
\theta^{(H)} S \equiv \pi^{(H)} = \pi^C - \pi^0
\]

(Smoothness of value function proven using methods in Pham [12])
Stochastic control problem for $p(t, s, y)$ when $Q^E = Q^M$

- Suppose $\lambda^S_t \equiv \lambda^S(t, S_t)$, and $\sigma^S_t \equiv \sigma^S(t, S_t)$
- Then infimum in (4) for $C = 0$ is achieved by $\psi = 0$, so

$$Q^E = Q^M$$

and $H^0(t, s, y) = H^E(t, s)$ is given by

$$H^E(t, s) = E_t^Q \left[ \frac{1}{2} \int_t^T (\lambda^S_u)^2 du \right]$$

Indifference price $p(t, s, y)$ has stochastic control representation

$$p(t, s, y) = \sup_{\psi \in \Psi} E_{t,s}^Q \left[ C(Y_T) - \frac{1}{2\alpha} \int_t^T \psi^2_u du \right]$$

subject to state dynamics

$$dS_t = \sigma^S(t, S_t) S_t dB_{t}^{S,Q}$$

$$dY_t = \sigma^Y_t Y_t \left[ (\lambda^Y_t - \rho_t \lambda^S_t - \sqrt{1 - \rho^2_t \psi_t}) dt + dB_{t}^{Y,Q} \right]$$
Indifference price PDE

- Define $\phi(t, s, y) = \sqrt{1 - \rho^2 \sigma^2 y p_y}$. HJB equation for $p(t, s, y)$ is

$$p_t + A^{QM}_{S,Y} p + \max_{\psi} \left[ -\frac{1}{2\alpha} \psi^2 - \phi \psi \right] = 0, \quad p(T, s, y) = C(y)$$

Optimal Markov control is $\psi^C_t \equiv \psi^C(t, S_t, Y_t)$, where

$$\psi^C(t, s, y) = -\alpha \phi(t, s, y)$$

Note that $\alpha \to 0 \Rightarrow \psi^C \to \psi^0 = 0$

- Hence $p$ solves semi-linear PDE

$$p_t + A^{QM}_{S,Y} p + \frac{1}{2} \alpha \phi^2 = 0, \quad p(T, s, y) = C(y)$$

For $\alpha \to 0$, obtain linear PDE for marginal price $p^M$, and

$$p^M(t, s, y) := \lim_{\alpha \to 0} p(t, s, y) = E^{QM}_{t,s,y} C(Y_T)$$
Hedging error (residual risk) process

\[ R_t := p(0, S_0, Y_0) + \int_0^t \theta_u^{(H)} dS_u - p(t, S_t, Y_t), \quad R_0 = 0, \quad 0 \leq t \leq T \]

Itô and PDE for \( p(t, s, y) \) gives, with \( \phi_t \equiv \phi(t, S_t, Y_t) \),

\[ dR_t = \frac{1}{2} \alpha \phi_t^2 dt - \phi_t dZ_t^S \]

Define

\[ A_t := -\exp(-\alpha R_t), \quad 0 \leq t \leq T \]

Then

\[ A_t = -\mathcal{E}(\alpha \phi \cdot Z_t^S), \quad 0 \leq t \leq T \]

so \( A \) is a \( Q^M \)-martingale (and also a \( P \)-martingale)
Payoff decomposition and price representation

Integrate SDE for \( R \) over \([t, T]\) and use definition of \( R \) (equivalently, directly use Itô and PDE for \( p(t, s, y) \) over \([t, T]\)):

\[
C(Y_T) = p(t, S_t, Y_t) - \frac{1}{2} \alpha \int_t^T \phi_u^2 du + \int_t^T \phi_u dZ_u^S + \int_t^T \theta_u^{(H)} dS_u
\]  

(6)

Expectation under \( Q^M \) given \((S_t, Y_t) = (s, y)\) gives

Lemma

*Indifference price satisfies*

\[
p(t, s, y) = p^M(t, s, y) + \frac{1}{2} \alpha E_{t,s,y}^{Q^M} \int_t^T \phi^2(u, S_u, Y_u) du
\]  

(7)

with

\[
\phi(t, S_t, Y_t) = \sqrt{1 - \rho_t^2 \sigma_t^Y} Y_t p_y(t, S_t, Y_t), \quad 0 \leq t \leq T
\]
Föllmer-Schweizer-Sondermann decomposition

Let $\alpha \to 0$ in payoff decomposition (6) (or directly use PDE for $p^M$ and Itô) to obtain

$$C(Y_T) = p^M(t, S_t, Y_t) + \int_t^T \phi^M_u dZ_u^S + \int_t^T \theta^M_u dS_u$$

where

$$\theta^M_t := p^M_s(t, S_t, Y_t) + \rho_t \frac{\sigma^Y_t Y_t}{\sigma^S_t S_t} p^M_y(t, S_t, Y_t), \quad 0 \leq t \leq T$$

is the marginal hedging strategy and

$$\phi^M_t := \sqrt{1 - \rho_t^2 \sigma^Y_t Y_t p^M_y(t, S_t, Y_t)}, \quad 0 \leq t \leq T$$
Small risk aversion expansion

Denote

\[ \nu(t, s, y) := \text{var}^{QM}_{t, s, y}[C(Y_T)] \]

**Theorem**

The indifference pricing function \( p(t, s, y) \) has the asymptotic expansion

\[
p(t, s, y) = p^M(t, s, y) + \frac{1}{2} \alpha \left[ \nu(t, s, y) - E^{QM}_{t, s, y} \langle (\theta^M \cdot S) \rangle_{[t, T]} \right] + O(\alpha^2)
\]
Proof.

Write
\[ p(t, s, y) = \rho^*(t, s, y) + \alpha p^{(1)}(t, s, y) + O(\alpha^2) \]

Apply to indifference price representation (7) to obtain
\[ p^{(1)}(t, s, y) = \frac{1}{2} \mathbb{E}^{Q^M}_{t, s, y} \int_t^T (\phi^M_u)^2 \, du \]

But from FSS decomposition, compute
\[ \nu(t, s, y) = \mathbb{E}^{Q^M}_{t, s, y} \int_t^T \left[ (\phi^M_u)^2 + (\theta^M_u)^2 \right] \, du \]

and result follows

Higher order corrections computable
Invariance principle for Malliavin calculus

- Wiener space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\)
- \(\Omega = \mathcal{C}([0, T]), \omega : [0, T] \rightarrow \mathbb{R}, W_t(\omega) = \omega(t)\) is BM and \(P\) is Wiener measure

For continuous, bounded square-integrable process \(\varphi\), define

\[
\Phi := \int_0^t \varphi_u^2 du, \quad 0 \leq t \leq T
\]

For \(\epsilon \in \mathbb{R}\), define measure \(P^\epsilon\) on \((\Omega, \mathcal{F})\) by

\[
\frac{dP^\epsilon}{dP} = \Gamma^\epsilon := \mathcal{E}(\epsilon \varphi \cdot W)_T
\]

Under \(P^\epsilon\), \(W - \epsilon \Phi\) is BM, so for square-integrable Brownian functional \(F\) (\(\mathcal{F}_T\)-measurable mapping \(F : \Omega \rightarrow \mathbb{R}\) with \(\mathbb{E}F^2(W) < \infty\)), we have invariance principle

\[
\mathbb{E}F(W) = \mathbb{E}^\epsilon F(W - \epsilon \Phi) = \mathbb{E}[F(W - \epsilon \Phi) \Gamma^\epsilon]
\]

(8)

Malliavin calculus arises because we can differentiate (8) w.r.t. \(\epsilon\) at \(\epsilon = 0\).
Application to indifference price optimisation problem

Simplify notation: take \( t = 0 \), relabel \( Q \rightarrow P \)

\[
p = \sup_{\psi} E \left[ C(Y_T) - \frac{1}{2\alpha} \int_0^T \psi^2_t \, dt \right]
\]

subject to

\[
\begin{align*}
dY_t &= \sigma_t^Y Y_t \left[ \left( a_t - \sqrt{1 - \rho_t^2} \psi_t \right) \, dt + \rho_t dB_t + \sqrt{1 - \rho_t^2} dZ_t \right] \\
dS_t &= \sigma_t^S S_t dB_t
\end{align*}
\]

Idea is to consider \( \psi \) as a perturbation: write

\[
\epsilon \varphi_t := -\psi_t
\]

for some small parameter \( \epsilon \). Then

\[
\frac{\psi^2}{\alpha} = \frac{\epsilon^2}{\alpha} \varphi^2 = \varphi^2,
\]

if we choose \( \epsilon^2 := \alpha \).
Re-formulated problem

For simplicity, take $\sigma^S, \sigma^Y \rho$ constant. Then control problem is

$$p = \sup_{\varphi} E \left[ C(Y_T^\epsilon) - \frac{1}{2} \int_0^T \varphi_t^2 dt \right]$$

subject to

$$dY_t^\epsilon = \sigma^Y Y_t^\epsilon \left[ a(t, S_t, Y_t^\epsilon) dt + \rho dB_t + \sqrt{1 - \rho^2} (dZ_t + \epsilon \varphi_t dt) \right]$$

$$dS_t = \sigma^S S_t dB_t$$

where we write $Y^\epsilon$ to emphasise dependence on $\epsilon$
Invariance principle

Look for measure $P^\epsilon$ such that

$$\text{Law} \left( Y^\epsilon ; P \right) = \text{Law} \left( Y^0 ; P^\epsilon \right)$$

So define $P^\epsilon$ by

$$\frac{dP^\epsilon}{dP} = \mathcal{E}(\epsilon \phi \cdot Z)_T =: \Gamma^\epsilon$$

Then

$$\frac{1}{\epsilon} \left[ EC(Y^\epsilon_T) - EC(Y^0_T) \right] = \frac{1}{\epsilon} E \left[ (\Gamma^\epsilon - 1) C(Y^0_T) \right]$$  \hspace{1cm} (9)
Lemma

Function $\epsilon \to EC(Y^\epsilon_T)$ is differentiable at $\epsilon = 0$, with

$$
\frac{d}{d\epsilon} EC(Y^\epsilon_T)\big|_{\epsilon=0} = E \left[ C(Y^0_T) \int_0^T \varphi_t dZ_t \right]
$$

(10)

Proof.

Use

$$
\frac{1}{\epsilon} (\Gamma^\epsilon - 1) \to \int_0^T \varphi_t dZ_t, \quad \text{in } L_2 \text{ as } \epsilon \to 0
$$

(11)

Establishes (10) in view of (9) and $C$ bounded
Then, from (10)
\[
E \left[ C(Y_T^\varepsilon) - \frac{1}{2} \int_0^T \varphi_t^2 dt \right] = EC(Y_T^0) + E \left[ \varepsilon C(Y_T^0) \int_0^T \varphi_t dZ_t - \frac{1}{2} \int_0^T \varphi_t^2 dt \right] + \ldots
\]

Recall FSS decomposition, which in the current notation reads as
\[
C(Y_T^0) = EC(Y_T^0) + (\phi^M \cdot Z)_T + (\theta^M \cdot S)_T
\]

Then (12) becomes
\[
E \left[ C(Y_T^\varepsilon) - \frac{1}{2} \int_0^T \varphi_t^2 dt \right] = EC(Y_T^0) + E \left[ \int_0^T \left( \varepsilon \phi_t^M \varphi_t - \frac{1}{2} \varphi_t^2 \right) dt \right] + \ldots
\]

Maximise by choosing \( \varphi = \varepsilon \phi^M \), to obtain
\[
p = EC(Y_T^0) + \frac{1}{2} \varepsilon^2 E \int_0^T (\phi_t^M)^2 dt
\]

and result follows from FSS decomposition, since
\[
\text{var}[C(Y_T^0)] = E \int_0^T \left[ (\phi_t^M)^2 + (\theta_t^M)^2 \right] dt
\]
Lognormal model under partial information

- \( \sigma^S > 0, \sigma^Y > 0, \rho \in [-1, 1] \), known constants, inferred from \( \langle S \rangle, \langle Y \rangle, \langle S, Y \rangle \)
- \( \lambda^S, \lambda^Y \) are \( \mathcal{F}_0 \)-measurable random variables, so would be known constants if agent had access to filtration \( \mathbb{F} \)
- In full information (completely observable) model, hedger uses \( \mathbb{F} \)-adapted strategies
- In partial information model, hedger uses \( \hat{\mathbb{F}} \)-adapted strategies, where \( \hat{\mathbb{F}} \) is filtration generated by \( S, Y \)
Perfect correlation case

If correlation perfect $\rho = 1$, $\lambda^Y = \lambda^S$, and perfect hedge is

$$\Delta_t^{(BS)} = \frac{\sigma^Y}{\sigma^S} \frac{Y_t}{S_t} \frac{\partial}{\partial y} BS(t, Y_t; \sigma^Y), \quad 0 \leq t \leq T$$

BS-style hedge does not require knowledge of $\lambda^S, \lambda^Y$
Completely observable incomplete case

If $\rho \neq 1$, indifference price at $t \in [0, T]$, is $p^F(t, Y_t)$, where

$$p^F(t, y) = \frac{1}{\alpha(1 - \rho^2)} \log E^{Q^M} \left[ \exp (\alpha(1 - \rho^2)C(Y_T)) \right| Y_t = y]$$

(13)

Under $Q^M$, $Y$ follows

$$dY_t = \sigma^Y Y_t [(\lambda^Y - \rho \lambda^S) dt + dB_t^{Y, Q^M}$$

Optimal hedging strategy $\Delta^F_t$ given by

$$\Delta^F_t = \rho \frac{\sigma^Y Y_t}{\sigma^S S_t} \frac{\partial p^F}{\partial y}(t, Y_t), \quad 0 \leq t \leq T$$

Asymptotic expansion of (13) in powers of $\epsilon := \alpha(1 - \rho^2)$

$$p^F(t, y) = E^{Q^M} [h(Y_T) | Y_t = y] + \frac{1}{2} \epsilon \var^Q \left[ h(Y_T) | Y_t = y \right] + O(\epsilon^2)$$

Requires knowledge of $\lambda^S, \lambda^Y$
Partial information case

\( \lambda^S, \lambda^Y \) are random variables with some prior distribution

\[
\xi^S_t := \frac{1}{\sigma^S} \int_0^t \frac{dS_u}{S_u} = \lambda^S t + B^S_t, \quad \xi^Y_t := \frac{1}{\sigma^Y} \int_0^t \frac{dY_u}{Y_u} = \lambda^Y t + B^Y_t
\]

or

\[
\xi^S_t = \frac{1}{\sigma^S} \log \left( \frac{S_t}{S_0} \right) + \frac{1}{2} \sigma^S t, \quad \xi^Y_t = \frac{1}{\sigma^Y} \log \left( \frac{Y_t}{Y_0} \right) + \frac{1}{2} \sigma^Y t
\]

Consider

\[
\Xi_t := \begin{pmatrix} \xi^S_t \\ \xi^Y_t \end{pmatrix}, \quad 0 \leq t \leq T,
\]

as observation process in Kalman-Bucy filter: noisy observations of “signal process”

\[
\Lambda := \begin{pmatrix} \lambda^S \\ \lambda^Y \end{pmatrix}
\]
Prior

Observation filtration  \( \hat{F} := (\hat{F}_t)_{0 \leq t \leq T} \)

\[ \hat{F}_t = \sigma(\xi_u^S, \xi_u^Y; 0 \leq u \leq t), \quad 0 \leq t \leq T \]

Assume Gaussian prior:

\[ \text{Law}(\Lambda | \hat{F}_0) = N(\Lambda_0, V_0) \]

\[ \Lambda_0 = \begin{pmatrix} \lambda_0^S \\ \lambda_0^Y \end{pmatrix}, \quad V_0 = \begin{pmatrix} v_0^S & c_0 \\ c_0 & v_0^Y \end{pmatrix}, \quad c_0 := \rho \min(v_0^S, v_0^Y) \] (14)

- Motivation: agent uses data before time zero to make point estimate of \( \Lambda \), and uses distribution of estimator as prior
- With historical data for \( \xi^S (\xi^Y) \) over interval \( t_S (t_Y) \), then unbiased estimator of \( \Lambda \) is Gaussian according to (14) with \( \lambda_0^i = \lambda^i \), and \( v_0^i = 1/t_i \), for \( i = S, Y \)
Observation and signal SDEs

\[ d\Xi_t = \Lambda dt + D d\mathbf{B}_t, \quad d\Lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

where

\[ D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \quad \mathbf{B}_t = \begin{pmatrix} B^S_t \\ Z^S_t \end{pmatrix} \]

Optimal filter \( \hat{\Lambda}_t := E[\Lambda|\hat{\mathcal{F}}_t], 0 \leq t \leq T \), is two conditional expectations

\[ \hat{\lambda}^i_t := E[\lambda^i|\hat{\mathcal{F}}_t], \quad 0 \leq t \leq T, \quad i = S, Y \]

Conditional variances and covariance

\[ v^i_t := E \left[ (\lambda^i - \hat{\lambda}^i_t)^2 \Big| \hat{\mathcal{F}}_t \right], \quad 0 \leq t \leq T, \quad i = S, Y \]

\[ c_t := E \left[ (\lambda^S - \hat{\lambda}^S_t)(\lambda^Y - \hat{\lambda}^Y_t) \Big| \hat{\mathcal{F}}_t \right], \quad 0 \leq t \leq T \]
Covariance matrix

\[ V_t := \begin{pmatrix} v_t^S & c_t \\ c_t & v_t^Y \end{pmatrix}, \quad 0 \leq t \leq T \]

Well known that \((V_t)_{0 \leq t \leq T}\) is deterministic. Introduce notation

\[ m_t := \min(v_t^S, v_t^Y), \quad M_t := \max(v_t^S, v_t^Y), \quad b_t := \frac{M_t - \rho^2 m_t}{1 - \rho^2} \]

Kalman-Bucy filter to converts the partial information model to an equivalent full information model
Proposition

Effective full information model on $\left( \Omega, \mathcal{F}_T, \mathbb{F}, P \right)$

$$
\begin{align*}
dS_t &= \sigma^S S_t (\hat{\lambda}_t^S dt + d\hat{B}_t^S), \\
dY_t &= \sigma^Y Y_t (\hat{\lambda}_t^Y dt + d\hat{B}_t^Y)
\end{align*}
$$

For $i, j \in \{ S, Y \}$, if $m_0 = v_0^i < v_0^j = M_0$, then

$$
\begin{align*}
\hat{\lambda}_t^i &= \frac{\lambda_0^i + m_0 \xi_t^i}{1 + m_0 t}, \\
\hat{\lambda}_t^j - \rho \hat{\lambda}_t^i &= \frac{\hat{\lambda}_0^j - \rho \hat{\lambda}_0^i + b_0 (\xi_t^j - \rho \xi_t^i)}{1 + b_0 t}
\end{align*}
$$

If $m_0 = v_0^S = v_0^Y = M_0$, then

$$
\begin{align*}
\hat{\lambda}_t^i &= \frac{\lambda_0^i + m_0 \xi_t^i}{1 + m_0 t}, \quad i = S, Y
\end{align*}
$$
Proposition

Functions $v^S, v^Y, c$ given by

\[ v^i_t = m_t, \quad v^j_t = M_t, \quad c_t = \rho m_t, \quad \text{if } m_0 = v^i_0 < v^j_0 = M_0, \quad i, j \in \{S, Y\} \]  \quad (18)

and

\[ v^S_t = v^Y_t = m_t = M_t, \quad c_t = \rho m_t, \quad \text{if } m_0 = v^S_0 = v^Y_0 = M_0 \]  \quad (19)

with

\[ m_t = \frac{m_0}{1 + m_0 t}, \quad b_t = \frac{b_0}{1 + b_0 t}, \quad 0 \leq t \leq T \]  \quad (20)
Proof

By Kalman-Bucy filter, \( \hat{\Lambda} \) satisfies

\[
d\hat{\Lambda}_t = V_t (DD^T)^{-1} (d\Xi_t - \hat{\Lambda}_t dt) =: V_t (DD^T)^{-1} dN_t, \quad \hat{\Lambda}_0 = \Lambda_0
\]  

(21)

Innovations process \( N \)

\[
N_t := \Xi_t - \int_0^t \hat{\Lambda}_u du, \quad 0 \leq t \leq T
\]

is \( \mathcal{F} \)-Brownian motion:

\[
N_t = \begin{pmatrix} \hat{B}_t^S \\ \hat{B}_t^Y \end{pmatrix}, \quad \langle \hat{B}^S, \hat{B}^Y \rangle_t = \rho t, \quad 0 \leq t \leq T
\]

(22)

Using this and

\[
d\begin{pmatrix} S_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \sigma^S S_t \\ \sigma^Y Y_t \end{pmatrix} d\Xi_t
\]

gives dynamics (15) of \( S, Y \) in observation filtration
Proof, continued

Matrix $V_t$ satisfies Riccati equation

$$\frac{dV_t}{dt} = -V_t (DD^T)^{-1} V_t,$$

with $V_0$ given in (14). Then $F_t := V^{-1}_t$ satisfies Lyapunov equation

$$\frac{dF_t}{dt} = (DD^T)^{-1}$$

solved to give (18), (19) and (20). Use these in filtering equation (21) to get

1. For $i, j \in \{S, Y\}$, if $m_0 = v^i_0 < v^j_0 = M_0$,

$$\begin{align*}
    \hat{\lambda}^i_t &= m_t d\hat{B}^i_t = m_t (d\xi^i_t - \hat{\lambda}^i_t dt), \quad \hat{\lambda}_0^i = \lambda_0^i, \\
    d(\hat{\lambda}^j_t - \rho \hat{\lambda}^i_t) &= b_t (d\hat{B}^j_t - \rho d\hat{B}^i_t) = b_t [d(\xi^j_t - \rho \xi^i_t) - (\hat{\lambda}^j_t - \rho \hat{\lambda}^i_t) dt]
\end{align*}$$

with $\hat{\lambda}^j_0 = \lambda_0^j$

2. If $m_0 = v^S_0 = v^Y_0 = M_0$,

$$\begin{align*}
    \hat{\lambda}^i_t &= m_t d\hat{B}^i_t = m_t (d\xi^i_t - \hat{\lambda}^i_t dt), \quad \hat{\lambda}_0^i = \lambda_0^i, \quad i = S, Y
\end{align*}$$

From these SDEs obtain (16) and (17)
Abusing notation, we have

\[ \hat{\lambda}_t^S \equiv \hat{\lambda}^S(t, S_t), \quad \hat{\lambda}_t^Y \equiv \hat{\lambda}^Y(t, S_t, Y_t), \quad \text{if } v_0^S < v_0^Y \]
\[ \hat{\lambda}_t^S \equiv \hat{\lambda}^S(t, S_t), \quad \hat{\lambda}_t^Y \equiv \hat{\lambda}^Y(t, Y_t), \quad \text{if } v_0^S = v_0^Y \]
\[ \hat{\lambda}_t^S \equiv \hat{\lambda}^S(t, S_t, Y_t), \quad \hat{\lambda}_t^Y \equiv \hat{\lambda}^Y(t, Y_t), \quad \text{if } v_0^S > v_0^Y \]

(23)

\[ d\hat{\lambda}_t^S = m_t d\hat{B}_t^S, \quad d\hat{\lambda}_t^Y - \rho d\hat{\lambda}_t^S = b_t(d\hat{B}_t^Y - \rho d\hat{B}_t^S), \quad \text{if } v_0^S < v_0^Y \]
\[ d\hat{\lambda}_t^S = m_t d\hat{B}_t^S, \quad d\hat{\lambda}_t^Y = m_t d\hat{B}_t^Y, \quad \text{if } v_0^S = v_0^Y \]
\[ d\hat{\lambda}_t^Y = m_t d\hat{B}_t^Y, \quad d\hat{\lambda}_t^S - \rho d\hat{\lambda}_t^Y = b_t(d\hat{B}_t^S - \rho d\hat{B}_t^Y), \quad \text{if } v_0^S > v_0^Y \]

(24)

Intuition: estimation of drift of (geometric) Brownian motion depends only on length of time interval for which it is observed
Exponential hedging in effective full information model

- We have model with random drifts given by (23) and (24) (in limit $v^S \to 0, v^Y \to 0$, we recover standard full information model)
- So simply make replacements 
  
  \[ F \to \hat{F}, \quad B^i \to \hat{B}^i, \quad \lambda^i \to \hat{\lambda}^i, \quad i \in \{S, Y\} \]

  in Markovian basis risk model with random parameters
- For explicit results, consider case in which 
  
  \[ v_0^S \leq v_0^Y \iff \hat{\lambda}^S_t \equiv \hat{\lambda}^S(t, S_t) \]

  so that $Q^E = Q^M$, and take $\sigma^S, \sigma^Y, \rho$ constant
- Marginal price and hedge computable in closed form since, under $Q^M$, log $Y_T$ is Gaussian
Lognormal distribution for $Y_T$

**Proposition**

Under $Q^M$, conditional on $S_t = s$, $Y_t = y$, $\log Y_T \sim N(\mu, \Sigma^2)$, where $\mu \equiv \mu(t, s, y)$ and $\Sigma^2 \equiv \Sigma^2(t)$ are given by

$$mu(t, s, y) = \log y + \sigma^Y \left( \hat{\lambda}^Y(t, s, y) - \rho \hat{\lambda}^S(t, s) - \frac{1}{2} \sigma^Y \right) (T - t)$$

$$\Sigma^2(t) = \left[ 1 + (1 - \rho^2) b_t(T - t) \right] (\sigma^Y)^2 (T - t)$$

with $b_t = m_t$ if $v^S_0 = v^Y_0$

Gives BS-style formulae for marginal price and hedge. Higher order corrections easily computable too
Proof.

Use Itô and SDEs for $Y$ and $\lambda_t^Y - \rho \lambda_t^S$ under $Q^M$: for $t < T$,

$$
\log \frac{Y_T}{Y_t} = \sigma^Y \int_t^T (\lambda_u^Y - \rho \lambda_u^S) \, du - \frac{1}{2} (\sigma^Y)^2 (T - t) + \sigma^Y \int_t^T d\tilde{B}^Y_{u,Q^M} \tag{25}
$$

where $\tilde{B}^Y_{Q^M} = \rho \tilde{B}^S_{Q} + \sqrt{1 - \rho^2} \tilde{Z}^S$. Dynamics of $\lambda_t^Y - \rho \lambda_t^S$ under $Q^M$ are

$$
d(\lambda_t^Y - \rho \lambda_t^S) = \sqrt{1 - \rho^2} b_t d\tilde{Z}_t^S
$$

Hence, for $u > t$, after changing the order of integration in a double integral, we have

$$
\int_t^T (\lambda_u^Y - \rho \lambda_u^S) \, du = (\lambda_t^Y - \rho \lambda_t^S) (T - t) + \sqrt{1 - \rho^2} \int_t^T b_u (T - u) d\tilde{Z}_u^S
$$

Insert into (25) to yield result.
Numerical experiments

- Simulation study: generate asset price paths over $[-t_0, T]$ (so take $v_0^Y = v_0^S$)
- Use data over $[-t_0, 0]$ to estimate drifts, and so set prior at time 0
- Sell put at time 0 for $p^M(0, S_0, Y_0)$ and optimally hedge over $[0, T]$, incorporating updating from filtering
- Generate terminal hedging error, and repeat over many paths to generate terminal hedging error distribution
- Compare with BS-style hedge, and also with results in absence of filtering
Parameters

\[t_0 = 1, \quad \delta t = 1/504 \quad T = 1\]
\[S_{-t_0} = 80, \quad Y_{-t_0} = 80\]
\[\lambda^S = 0.3, \quad \sigma^S = 0.2, \quad \lambda^Y = 0.4, \quad \sigma^Y = 0.25, \quad \rho = 0.75\]
\[\alpha = 0.01, \quad K = 100\]
Using marginal price and associated hedge

Table: Hedging error statistics (as percentage of premium): \( \langle S_0 \rangle = 84.88, \langle Y_0 \rangle = 86.25, \langle p^M_0 \rangle = 19.98, \langle \theta^M_0 \rangle = -0.5885; \langle p^{BS}_0 \rangle = 19.96, \langle \Delta^{BS}_0 \rangle = -0.8397; \langle p^{NF}_0 \rangle = 19.75, \langle \Delta^{NF}_0 \rangle = -0.6284 \)

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With higher correlation, \( \rho = 0.9 \)

Table: Hedging error statistics, \( \rho = 0.9 \). \( \langle S_0 \rangle = 84.90, \langle Y_0 \rangle = 86.31, \langle p_0^M \rangle = 19.75, \langle \theta_0^M \rangle = -0.7325; \langle p_0^{BS} \rangle = 19.91, \langle \Delta_0^{BS} \rangle = -0.8414; \langle p_0^{NF} \rangle = 19.64, \langle \Delta_0^{NF} \rangle = -0.7547 

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Further questions

- General case, with random parameters and $Q^E \neq Q^M$
- Relationship with risk tolerance, and general representation for asymptotic expansions when $\text{dom}(U) = \mathbb{R}$
References


M. Monoyios, Marginal utility-based hedging of claims on non-traded assets with partial information. Submitted, 2008.

