Prospect Theory, Partial Liquidation and the Disposition Effect

Vicky Henderson
Oxford-Man Institute of Quantitative Finance
University of Oxford

vicky.henderson@oxford-man.ox.ac.uk

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The Problem

- Consider an agent with prospect theory preferences who seeks to liquidate a portfolio of (divisible) claims -
  * how does the agent sell-off claims over time?
  * how does prospect theory alter the agent’s strategy vs (rational) expected utility?
  * is the strategy consistent with observed behavior eg. disposition effect?
- Examples of claims might include stocks, executive stock options, real estate, managerial projects,...
Prospect Theory (Kahneman and Tversky (1979))

- In a rational world, agents evaluate risky gambles using expected utility (dating back to Von Neumann and Morgenstern (1944))
- Experimental work has showed substantial violations of expected utility theory
- Kahneman and Tversky (1979) proposed PT -
  * utility defined over gains and losses relative to a reference point, rather than final wealth (Markowitz (1952))
  * utility function exhibits concavity in the domain of gains and convexity in the domain of losses
  * steeper for losses than for gains, a feature known as loss aversion
  * non-linear probability transformation whereby small probabilities are overweighted (we will ignore)
• The agent has prospect theory preferences denoted by the function $U(z); \, z \in \mathbb{R}$

(I) Piecewise exponentials: (Kyle, Ou-Yang and Xiong (2006))

\[
U(z) = \begin{cases} 
\phi_1(1 - e^{-\gamma_1 z}) & z \geq 0 \\
\phi_2(e^{\gamma_2 z} - 1) & z < 0 
\end{cases} 
\tag{1}
\]

where $\phi_1, \phi_2, \gamma_1, \gamma_2 > 0$.

Assume $\phi_1 \gamma_1 < \phi_2 \gamma_2$ so that $U'(0-) > U'(0+)$

(II) Piecewise power: (Tversky and Kahneman (1992))

\[
U(z) = \begin{cases} 
z^{\alpha_1} & z \geq 0 \\
-\lambda(-z)^{\alpha_2} & z < 0 
\end{cases} 
\tag{2}
\]

where $\alpha_1, \alpha_2 \in (0, 1)$ and $\lambda > 1$.

Locally infinite risk aversion, $U'(0-) = U'(0+) = \infty$. 
Figure 1: The solid line represents the piecewise power $S$-shaped function with $\lambda = 2.25$ and $\alpha_1 = \alpha_2 = 0.88$ (parameters are those found experimentally by Tversky and Kahneman (1992)). The dashed line represents the piecewise exponential $S$-shaped function with parameters $\phi_1 = 1$, $\phi_2 = 2.25$ and $\gamma_1 = \gamma_2 = 2$. 
The Disposition Effect (Shefrin and Statman (1985))

- Many studies find that investors are reluctant to sell assets trading at a loss relative to the price at which they were purchased.
- For large datasets of share trades of individual investors, Odean (1998) (and others) “finds the proportion of gains realized is greater than the proportion of realized losses”
- Disposition effects have also been found in other markets - real estate (Genesove and Mayer (2001)), traded options (Poteshman and Serbin (2003)) and executive stock options (Heath, Huddart and Lang (1999))
- Reluctance of managers to abandon losing projects “throwing good money after bad” (Shefrin (2001))
Shefrin and Statman (1985)

- Prospect theory has long been recognized as one potential way of understanding the disposition effect

Borrow an *example* from Shefrin and Statman (1985) to illustrate. An investor bought a stock a month ago for $50 and it is currently trading at $40. Suppose either the stock will increase to $50 next period or decrease to $30, with equal probability. Choosing between:

**A.** sell the stock now and make a $10 loss

**B.** wait, and have a 50% chance of losing a further $10 but a 50% chance of breaking even.
Shefrin and Statman (1985) conclude that since the choice between the lotteries is associated with the convex portion of the \textit{S}-shaped function, the prospect theory investor would choose option B, thus waiting to gamble on the possibility of breaking even. They also recognize that this will depend on the odds of breaking even - and that if these were sufficiently unfavourable, the investor may choose lottery A, and sell for a loss today.
Related Literature

Kyle, Ou-Yang and Xiong (2006, JET)
Barberis and Xiong (2008a, JF)/Hens and Vlcek (2005)
Barberis and Xiong (2008b, preprint)
Kaustia (2008, JFQA)

Our Approach

• Optimal stopping model - covers egs of PT, time-homogeneous price process (recover Kyle et al (2006) as example) and easily extends to divisible positions
• Direct approach to optimal stopping (Dynkin (1965), Dayanik and Karatzas (2003)) - avoids lack of smooth-pasting
• In contrast to literature, we present a model where behavior consistent with eg. of Shefrin and Statman (1985) - eg. where sell at loss voluntarily, rather than only liquidating at loss if exogenously forced to do so
• Show results not robust to S-shaped function, or to divisibility
Price Dynamics

- Let $Y_t$ denote the asset price. Work on a filtration
  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting a BM $W = \{W_t, t \geq 0\}$ and assume $Y_t$
  follows a time-homogeneous diffusion process with state space
  $\mathcal{I} \subseteq \mathbb{R}$ and

  $$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \quad Y_0 = y_0$$

  with Borel functions $\mu: \mathcal{I} \to \mathbb{R}$ and $\sigma: \mathcal{I} \to (0, \infty)$.

  We assume $\mathcal{I}$ is an interval with endpoints $-\infty \leq a_\mathcal{I} < b_\mathcal{I} \leq \infty$ and
  that $Y$ is regular in $(a_\mathcal{I}, b_\mathcal{I})$.

  Denote $\tau^Y_{(a,b)} = \inf\{u : Y_u \notin (a, b)\}$. 

Definition 1 (Revuz and Yor (1999)) A locally bounded Borel function $s$ is a scale function if and only if the process $s(Y_{t \wedge \tau^Y_{(a_I,b_I)}}); t \geq 0$ is a local martingale. Furthermore, for arbitrary but fixed $c \in I$, we have $s(y) = \int_c^y \exp \left( - \int_c^x \frac{2\mu(z)}{\sigma^2(z)} dz \right) dx; \ y \in I$. $s(y)$ is real-valued, strictly increasing, and continuous on $I$. Finally, we have $\mathcal{A}s(.) = 0$ where the second order differential operator

$$\mathcal{A}u(y) := \frac{1}{2} \sigma^2(y) \frac{d^2 u}{dy^2}(y) + \mu(y) \frac{du}{dy}(y), \ \text{on } I$$

is the infinitesimal generator of $Y$. 
Price Dynamics - Examples

(I) EBM:
$Y$ follows $dY = Y(\mu dt + \sigma dW)$ for constants $\mu$ and $\sigma > 0$.
$I = (0, \infty)$. Define $\beta = 1 - 2\mu/\sigma^2$. (If $\beta < 0$ then $Y_t$ grows to $\infty$ whereas, if $\beta > 0$, then $Y_t$ tends to zero, almost surely.)
We have $s(y) = y^\beta$ if $\beta > 0$ and $s(y) = -(y)^\beta$ if $\beta < 0$.

(II) BM:
$Y$ follows $dY = \mu dt + \sigma dW$, again for constants $\mu$ and $\sigma > 0$.
$I = (-\infty, \infty)$.
We have $s(y) = -e^{-\frac{2\mu}{\sigma^2} y}$ if $\mu > 0$ and $s(y) = e^{-\frac{2\mu}{\sigma^2} y}$ if $\mu < 0$. 
The Optimal Stopping Problem - Indivisible Claims

- Agent chooses when to receive payoff $h(Y_\tau)$, $h$ non-decreasing. Let $h_R$ denote the reference level. Interpret $h_R$ as price paid, hence “breakeven” level.
- Agent’s objective is:

$$V_1(y) = \sup_{\tau} \mathbb{E}[U(h(Y_\tau) - h_R)|Y_0 = y], \quad y \in \mathcal{I}$$

(3)

where $U(.)$ is increasing

- Assume a zero interest or discount rate. Aids comparison to Kyle et al (2006) (and Barberis and Xiong (2008a)). In contrast, in Barberis and Xiong (2008b), a positive discount rate is important in giving the investor an incentive to realize gains today and delay losses (indefinitely)- abstract from such an incentive
**Heuristics**

- Approach is to consider stopping times of the form “stop when price $Y$ exits an interval” and choose the “best” interval.
- The key is to transform into natural scale via $\Theta_t = s(Y_t)$. Let $\Theta_0 = \theta_0 = s(y_0)$. Recall from Definition, the scale function $s(.)$ is such that the scaled price $\Theta_t$ is a (local) martingale.

Then

$$
\tau_{(a,b)}^Y := \inf \{ u : Y_u \not\in (a, b) \} \equiv \inf \{ u : \Theta_u \not\in (s(a), s(b)) \}
$$

$$
= \inf \{ u : \Theta_u \not\in (\phi, \psi) \} := \tau_{(\phi, \psi)}^\Theta
$$

where we define $\phi = s(a)$, $\psi = s(b)$.
Define $f(y) = U(h(y) - h_R)$, and

$$g_1(\theta) = f(s^{-1}(\theta)) := U(h(s^{-1}(\theta)) - h_R) \quad (4)$$

Note $g_1(\theta)$ increasing in $\theta$.

$g_1(\theta)$ represents the value of the game if the unit of claim is sold immediately.

Then, for any fixed interval $(a, b) \in \mathcal{I}$ such that $(s(a), s(b))$ is a bounded interval,

$$\mathbb{E}[f(Y_{\tau_{(a,b)}})|Y_0 = y] = \mathbb{E}[f(s^{-1}(\Theta_{\tau_{(\phi,\psi)}}))|\Theta_0 = \theta]$$

$$= \mathbb{E}[g_1(\Theta_{\tau_{(\phi,\psi)}})|\Theta_0 = \theta] = g_1(\phi)\frac{\psi - \theta}{\psi - \phi} + g_1(\psi)\frac{\theta - \phi}{\psi - \phi}$$

Then

$$\sup_{\phi < \theta < \psi} \left\{ g_1(\phi)\frac{\psi - \theta}{\psi - \phi} + g_1(\psi)\frac{\theta - \phi}{\psi - \phi} \right\} = \bar{g}_1(\theta)$$

to which the solution is given by taking the smallest concave majorant $\bar{g}_1(\theta)$ of $g_1(\theta)$. 

Figure 2: *Stylized representation of the function $g_1(\theta)$ as a function of transformed price $\theta$, where $\theta = s(y)$.***
Proposition 2  On the interval \((s(a_I), s(b_I))\), let \(\bar{g}_1(\theta)\) be the smallest concave majorant of \(g_1(\theta) := f(s^{-1}(\theta))\).

(i) Suppose \(s(a_I) = -\infty\). Then

\[ V_1(y) = f(b_I) = U(h(b_I) - h_R); \quad y \in (a_I, b_I) \]

(ii) Suppose \(s(a_I) > -\infty\). Then

\[ V_1(y) = \bar{g}_1(s^{-1}(y)); \quad y \in (a_I, b_I) \]

Proof:
Although this follows from Dynkin (1965), and more recently, Dayanik and Karatzas (2003)) it is straightforward to prove the result directly here

(i) Trivially \(V_1(y) \leq f(b_I)\). Let \(b_n \uparrow b_I\) and let \(\tau_n = \tau^Y_{(a_I, b_n)}\). Then \(V_1(y) \geq f(b_n) \uparrow f(b_I)\).
(ii) By definition,

$$V_1(y) = \sup_{\tau} \mathbb{E}[f(Y_{\tau})|Y_0 = y] = \sup_{\tau} \mathbb{E}[g_1(\Theta_{\tau})|\Theta_0 = \theta]$$

But

$$\mathbb{E}[g_1(\Theta_{\tau})|\Theta_0 = \theta] \leq \mathbb{E}[\bar{g}_1(\Theta_{\tau})|\Theta_0 = \theta] \leq \bar{g}_1(\mathbb{E}[\Theta_{\tau}|\Theta_0 = \theta])$$

where we use the fact $\bar{g}_1$ is the smallest concave majorant of $g_1$ and Jensen’s inequality. Finally we use that $\bar{g}_1$ is increasing, and that a local martingale bounded below is a supermartingale to give

$$\bar{g}_1(\mathbb{E}[\Theta_{\tau}|\Theta_0 = \theta]) \leq \bar{g}_1(\theta)$$

and hence $V_1(y) \leq \bar{g}_1(\theta)$.  

16
It remains to show that there is a stopping rule which attains this bound. Let \( \Upsilon = \{ \upsilon : \bar{g}_1(\upsilon) = g_1(\upsilon) \} \), and given \( \theta \), choose

\[
\phi^* = \sup\{ \xi < \theta : \xi \in \Upsilon \}, \\
\psi^* = \inf\{ \xi > \theta : \xi \in \Upsilon \}.
\]

Then \( \bar{g}_1(\theta) \) is linear on the interval \( \theta \in (\phi^*, \psi^*) \).

If \( \psi^* < \infty \) (eg. if \( s(b_I) < \infty \)), then

\[
\mathbb{E}[f(Y_{\tau^*_{\phi^*,\psi^*}}) | \Theta_0 = \theta] = \mathbb{E}[g_1(\Theta_{\tau^*_{\phi^*,\psi^*}})] = \mathbb{E}[\bar{g}_1(\Theta_{\tau^*_{\phi^*,\psi^*}})] = \bar{g}_1(\theta).
\]

If \( \psi^* = \infty \), then use \( \tau^* = \tau^\Theta_{\phi^*,\psi_n} = \tau^Y_{s^{-1}(\phi^*,s^{-1}(\psi_n))} \) and take limits \( \psi_n \to \infty \). \( \square \)
Example 1: Piecewise Exponential $S$-shaped utility and Brownian motion (cf. Kyle, Ou-Yang, Xiong (2006))

Proposition 3 The solution to problem (3) with $h(y) = y$, when $Y$ follows BM and $U(z)$ is given by piecewise exponential $S$-shape, consists of four cases:

(I): If $\mu \geq 0$, the agent waits indefinitely (see Figure 3(a) and 3(b)).

(II) If $\mu < 0$ and $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$ and $|\mu|/\sigma^2 < \frac{1}{2} \frac{\phi_1}{\phi_2} \gamma_1$, the agent stops at and above a level $\bar{y}_u^{(1)} > y_R$ given by:

$$\bar{y}_u^{(1)} = y_R - \frac{1}{\gamma_1} \ln \left( \left( \frac{2\mu}{2\mu - \gamma_1 \sigma^2} \right) \left( \frac{\phi_1 + \phi_2}{\phi_1} \right) \right)$$

(see Figure 4(a)).

(III) If $\mu < 0$ and $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$ and $|\mu|/\sigma^2 \geq \frac{1}{2} \frac{\phi_1}{\phi_2} \gamma_1$, the agent stops everywhere at and above the break-even point $y_R$, but waits below the break-even point. Thus if the agent sells, she exactly breaks even (see Figure 5(a)).

(IV) If $\mu/\sigma^2 \leq -\frac{1}{2}\gamma_2$, the agent sells immediately at all price levels (see Figure 6(a)).
(a) (I). \( \mu = 0.3, (\mu/\sigma^2 > \frac{1}{2}\gamma_1) \).
The agent waits everywhere.

(b) (I). \( \mu = 0.1, (\mu/\sigma^2 < \frac{1}{2}\gamma_1) \).
The agent waits everywhere.

Figure 3: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: \( \sigma = 0.4, \phi_1 = 0.2, \phi_2 = 1, \gamma_1 = 3, \gamma_2 = 1 \) and reference level, \( y_R = 1 \).
(a) (II). $\mu = -0.03$, $(|\mu|/\sigma^2 < \frac{1}{2} \phi_1 \gamma_1)$. $s(y_R) = 1.455$. The agent stops for $\theta > 1.54$; equivalently, for prices $y > 1.15$.

Figure 4: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$. 

\[
geq
\]
Figure 5: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$. 

(a) (III). $\mu = -0.06$. 

$|\mu|/\sigma^2 > \frac{1}{2} \frac{\phi_1}{\phi_2} \gamma_1$).

$s(y_R) = 2.12$. The agent stops for $\theta \geq s(y_R)$, or equivalently, for prices $y \geq y_R = 1$. 

Figure 6: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$. Stop immediately.
• Non-trivial cases are (II) and (III) - either agent sells at break-even (and thus wouldn’t hold asset ex-ante) or gambles on selling at a gain

Comparison to Kyle, Ou-Yang and Xiong (2006)
• Kyle et al (2006) study the liquidation problem for an indivisible asset, with BM price and piecewise exponential S-shaped utility using variational techniques - non-differentiability implies cannot use smooth-pasting
• They rule out case (II) where agent liquidates at a gain
• They relate to disposition effect - but agent never chooses to sell at a loss - recall example from Shefrin and Statman (1985)
• Instead, behavior is focused on “selling at break-even” (ex-ante?)
Example 2: Piecewise Power $S$-shaped utility and Exponential BM

Proposition 4 The solution to problem (3) with $h(y) = y$, when $Y$ follows Exponential BM and $U(z)$ is given by piecewise power $S$-shape, consists of three cases. Recall $\beta = 1 - \frac{2\mu}{\sigma^2}$.

(I): If $\beta \leq 0$; or if $0 < \beta < \alpha_1 < 1$, the agent waits indefinitely and never liquidates (see Figure 7(a) and 7(b)).

(II) If $0 < \alpha_1 < \beta \leq 1$ or $\alpha_1 = \beta < 1$, the agent stops at a level higher than the break-even point. If the agent liquidates, she does so at a gain (see Figure 8(a)).

(III) If $\beta > 1$, the agent stops when the price reaches either of two levels. These two levels are on either side of the break-even point - liquidates either at a gain or at a loss (see Figure 8(b)).
Figure 7: Optimal Liquidation of an Indivisible Asset under Power S-shaped utility and Exponential Brownian motion price process. Common parameters are: $\lambda = 2.2$, $\alpha_2 = \alpha_1$ and reference level $y_R = 1$.
(a) (II). $\beta = 0.75$, $\alpha_1 = 0.5 < \beta$, $s(y_R) = 1$. The agent stops for $\theta \geq \theta_u^{(1)} = 1.06$ or equivalently for $y \geq \bar{y}_u^{(1)} = 1.08$.

(b) (III). $\beta = 1.5$, $\alpha_1 = 0.7$, $s(y_R) = 1$. The agent waits for $\theta \in (\theta_l^{(1)} = 0.1723, \theta_u^{(1)} = 1.0105)$ and stops otherwise. Equivalently, the agent waits for $y \in (\bar{y}_l^{(1)} = 0.31, \bar{y}_u^{(1)} = 1.007)$.

Figure 8: Optimal Liquidation of an Indivisible Asset under Power S-shaped utility and Exponential Brownian motion price process. Common parameters are: $\lambda = 2.2$, $\alpha_2 = \alpha_1$ and reference level $y_R = 1$. 
Remarks - Piecewise Power functions

- Conclusions (and findings of Kyle et al) not robust to changing the S-shaped function - in case (III) the agent sells at a loss - and here, never stop at the breakeven
- Piecewise power functions lead to situation where if odds are bad enough (price transient to zero, a.s), agent “gives up” and sells at a loss - consistent with eg. of Shefrin and Statman (1985)
- but, agent would take the position ex-ante... (cf. Hens and Vlcek (2005), Barberis and Xiong (2008a) and Kaustia (2008))
Proposition 5 For $\beta > 1$ (case (III) of Proposition 4), there are two selling thresholds either side of the breakeven point, denoted $\bar{y}_{u}^{(1)} > y_R$ and $\bar{y}_{l}^{(1)} < y_R$. If $\alpha_2 = \alpha_1$, rewrite as:

$$\bar{y}_{u}^{(1)} = \bar{c}_u y_R \quad \text{and} \quad \bar{y}_{l}^{(1)} = \bar{c}_l y_R$$

for constants $\bar{c}_l < \bar{c}_u$ with $\bar{c}_l < 1, \bar{c}_u > 1$, where:

$$\frac{\alpha_1}{\beta} (\bar{c}_u - 1)^{\alpha_1 - 1} \bar{c}_u^{1-\beta} = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda (1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}}$$  \hspace{1cm} (5)

$$\frac{\lambda \alpha_1}{\beta} (1 - \bar{c}_l)^{\alpha_1 - 1} \bar{c}_l^{1-\beta} = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda (1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}}.$$ \hspace{1cm} (6)

For $\alpha_1 < \beta$ and $0 < \beta < 1$ (case (II) of Proposition 4), there is a single selling threshold above the breakeven point, denoted $\bar{y}_{u}^{(1)} > y_R$. If $\alpha_2 = \alpha_1$, then $\bar{y}_{u}^{(1)} = \bar{c}_u y_R$ where $\bar{c}_u$ solves (5) with $\bar{c}_l = 0$. 


Probability of Selling at a Gain - Disposition Effect

Suppose the agent has paid an amount $y_R$ for the asset and $y = y_R$. For $\beta > 1$ (case (III)), the probability of selling at a gain is given by:

$$\theta - \bar{\theta}_l^{(1)} \over \bar{\theta}_u^{(1)} - \bar{\theta}_l^{(1)} = 1 - (\bar{c}_l)^\beta$$

(7)

For $0 < \alpha_1 < \beta < 1$ (case (II)), this simplifies to $(\bar{c}_u)^{-\beta}$.

We see probability of selling at a gain (relative to a loss) is very high - consistent with disposition effect.
(a) Loss aversion parameter $\lambda = 2.2$

(b) Loss aversion parameter $\lambda = 1$

Figure 9: Probability of liquidating at a gain in Case (III), as a function of $\beta$ and $\alpha_1$. The reference level is $y_R = 1$ and take $y = 1$. 
Divisible Claims

Consider agent with \( n \geq 1 \) units of claim and initial wealth \( x \)

The agent’s objective is:

\[
V_n(y, x) = \sup_{\tau^n \leq \ldots \leq \tau^1} \mathbb{E}[U(x + \sum_{i=1}^{n} h(Y_{\tau^i}) - nh_R)|Y_0 = y] \tag{8}
\]

The agent compares the total payoff to the total reference level for \( n \) units, given by \( nh_R \).

Using conditioning, the value of the game for the agent with \( n \geq 1 \) units remaining can be re-expressed as

\[
V_n(y, x) = \sup_{\tau^n} \mathbb{E}[V_{n-1}(Y_{\tau^n}, x + h(Y_{\tau^n}) - h_R)|Y_0 = y]
\]

where define \( V_0(y, x) = U(x) \).
Define \( g_n(\theta, x) \) to be the value of the game with \( n \) units remaining, \( n \) reference levels, initial wealth \( x \) and plan to sell one unit immediately. Then

\[
g_n(\theta, x) = V_{n-1}(s^{-1}(\theta), x + h(s^{-1}(\theta)) - h_R)
\]

Then

\[
V_n(y, x) = \sup_{\tau^n} \mathbb{E}[V_{n-1}(Y_{\tau^n}, x + h(Y_{\tau^n}) - h_R)] = \sup_{\tau^n} \mathbb{E}[g_n(\Theta_{\tau^n}, x)]
\]

\[
= \sup_{\theta < \phi < \psi} \left\{ g_n(\phi, x) \frac{\psi - \theta}{\psi - \phi} + g_n(\psi, x) \frac{\theta - \phi}{\psi - \phi} \right\} = \bar{g}_n(\theta, x)
\]

where \( \bar{g}_n(\theta, x) \) is the smallest concave majorant of \( g_n(\theta, x) \). Hence

\[
g_n(\theta, x) = \bar{g}_{n-1}(\theta, x + h(s^{-1}(\theta)) - h_R)
\]

**Proposition 6** The solution to problem (8) with two units of asset when the asset price $Y$ follows Brownian motion and $U(z)$ is given by piecewise exponential $S$-shape in (1) consists of four cases:

(I): If $\mu \geq 0$, the agent waits indefinitely.

(II)/(III): If $\mu < 0$ and $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$, the agent sells both units at and above a level $\bar{y}_u^{(2)}$ which is itself greater than the break-even point, $y_R$. That is, the agent sells both units at a gain. The threshold level $\bar{y}_u^{(2)}$ is given by

$$
\bar{y}_u^{(2)} = \frac{1}{2} (2y_R - x) - \frac{1}{2\gamma_1} \ln \left( \left( \frac{2\mu}{2\mu - 2\gamma_1\sigma^2} \right) \left( \frac{\phi_1 + \phi_2}{\phi_1} \right) \right)
$$

(IV) If $\mu/\sigma^2 \leq -\frac{1}{2}\gamma_2$, the agent sells immediately at all price levels.
• Agent willing to gamble on larger risky stake \((n = 2)\) when expected return is poor, but not willing to enter ex-ante for smaller stake \((n = 1)\) (Case (III), Prop 3) - behaving as if convex utility - recall sell close to break-even so majority of region of interest is where function is \textit{convex}

• Break-even plays little role - finding in Kyle et al (2006) that “sell at break-even” is not robust to divisibility

• “All-or-nothing” sales strategy
Example: Piecewise power $S$-shaped utility and Exponential BM

**Proposition 7** The solution to problem (8) with two units of asset when the asset price $Y$ follows Exponential Brownian motion and $U(z)$ is given by piecewise power $S$-shape in (2) with $\alpha_2 = \alpha_1$, consists of three cases. Recall $\beta = 1 - \frac{2\mu}{\sigma^2}$.

*Case (I):* If $\beta \geq 0$; or if $0 < \beta < \alpha_1 < 1$, the agent waits indefinitely and never liquidates.

*Case (II):* If $0 < \alpha_1 < \beta \leq 1$ or $\alpha_1 = \beta < 1$, the agent waits in the region $\theta < \bar{\theta}_u^{(2)}$ and sells both units of asset in the region $\theta > \bar{\theta}_u^{(2)}$.

There are no asset values for which the agent sells a single unit of asset.

*Case (III):* If $\beta > 1$, the agent sells both units of asset at either of two levels $\bar{\theta}_l^{(2)}, \bar{\theta}_u^{(2)}$ on either side of the break-even point. There are no asset values for which the agent sells a single unit of asset.
• Contrast these results with those of an agent with standard concave utility (over wealth) where units are sold-off over time (cf. finitely divisible model of Grasselli and Henderson (2006), Rogers and Scheinkman (2007), or (infinitely divisible) Henderson and Hobson (2008))
• Consistent with the disposition effect - Odean (1998) shows that the disposition effect remains strong even when the sample is limited to sales of investor’s entire holdings of stock
Concluding Remarks

- Direct approach enables us to compare various specifications of prospect theory and price process and show results are not robust to $S$-shaped function or to the generalization to divisible positions.
- In contrast to existing literature, we provide prospect theory optimal stopping model (with Tversky and Kahneman (1992) piecewise power functions) under which the agent will liquidate (voluntarily) at a loss, enter the position ex-ante, and will be more likely to sell at a (small) gain than a (large) loss, consistent with disposition effect.
- We extend to divisible positions and show prospect agent prefers to liquidate on an “all-or-nothing” basis.