Portfolio Choice via Quantiles

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Based on the joint work with Prof Xunyu Zhou
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- Yaari’s “dual theory of choice” [Yaari (1987)]
- Kahneman and Tversky’s prospect theory [Kahneman and Tversky (1979), Tversky and Kahneman (1992)]
- Lopes’ SP/A theory [Lopes (1987) and Lopes and Oden (1999)]
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- Goal achieving problem [Kulldorff (1993), Heath (1993) and Browne (1999)]
- VaR/CVaR [Rockafellar and Uryasev (2000)]
- Law-invariant coherent risk measure [Artzner, Delbaen, Eber and Heath (1999) and Kusuoka (2001)]
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In this work, we propose a new framework to accommodate most of the aforementioned preferences and develop a new technique to solve the model.
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• Continuous time market
• Tame portfolios
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Continuous time market

Tame portfolios

Arbitrage-free and complete market

Dynamic portfolio selection can be translated into a static problem of choosing the optimal terminal payoff

“The more money the better”
A Non-Expected Utility Maximisation Model

We consider the following portfolio selection problem

\[
\begin{align*}
\text{Max}_X & \quad V(X) := \int_{-\infty}^{\infty} u(x) d \left[ -T(1 - F_X(x)) \right] \\
\text{Subject to} & \quad F_X(\cdot) \in \mathcal{F} \cap \mathcal{D}, \\
& \quad E[\rho X] \leq x_0
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\mathcal{F} = \{F(\cdot) : \mathbb{R} \rightarrow [0, 1] \mid F(\cdot) \text{ is increasing, càdlàg and } F(c) = 0 \text{ for some } c \in \mathbb{R}\}
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- \(\mathcal{D}\) is a subset of \(\mathcal{F}\), specifying the constraints imposed on the terminal payoff
- Both preference and constraints (other than the initial budget constraint) are law-invariant
Examples

- Expected Utility:

\[ \int_0^\infty u(x)dF_X(x) \]
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  \[ \int_0^\infty 1_{\{x \geq b\}} \, dF_X(x) \]
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- **SP/A:**
  \[ \int_{0}^{\infty} xd [-T (1 - F_X(x))] \]

- **Prospect Theory:**
  \[ \int_{B}^{\infty} u_+(x - B) d [-T_+ (1 - F_X(x))] - \int_{-\infty}^{B} u_- (B - x) d [T_- (F_X(x))] \]
The portfolio selection problem

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Change the Decision Variable

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Change of variable \( z = F_X(x) \)

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\int_{-\infty}^{\infty} u(x) d\left[-T(1 - F_X(x))\right] = \int_{0}^{1} u\left(F_X^{-1}(z)\right) T'(1 - z) dz
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= E\left[u(F_X^{-1}(Z))T'(1 - Z)\right]
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where \( Z \sim U(0, 1) \)
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If we regard \( F_X^{-1}(\cdot) \) as the variable, the distortion function is separated and we restore the concavity if \( u(\cdot) \) is concave
This suggests that it is better to regard the *quantile function* $F_X^{-1}(\cdot)$ as the decision variable. It works in the objective function
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It also works in the constraints

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F_X(\cdot) \in F \cap D \iff F_X^{-1}(\cdot) \in G \cap M
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where

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G := \{G(\cdot) : (0, 1) \to \mathbb{R} \mid G(\cdot) \text{ is increasing, càdlàg and } G(0+) > -\infty\}
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It, however, does not work in the budget constraint

$$E[\rho X] \leq x_0$$
A dual argument is applied to dealing with the budget constraint
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The performance only depends on the distribution of the terminal payoff, thus the dual problem is to minimise the cost of replicating the terminal payoffs following a given distribution
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Given a distribution function $F(\cdot)$, formulate the following dual problem

\[
\text{Min} \quad E[\rho X] \\
\text{Subject to} \quad X \text{ is } F(\cdot) \text{ distributed}
\]
Lemma (Jin and Zhou 2008)

If $\rho$ has no atom, then $Z := 1 - F_\rho(\rho)$ is uniformly distributed and $E[\rho F^{-1}(Z)] \leq E[\rho X]$ for any $F(\cdot)$ distributed r.v. $X$. Moreover, if $E[\rho F^{-1}(Z)] < \infty$, then the inequality is equality iff $X = F^{-1}(Z)$.

- $X = F^{-1}(Z) = F^{-1}(1 - F_\rho(\rho))$, where $Z \sim U(0,1)$, uniquely solves the dual problem.
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- The assumption \( \rho \) is atomless is crucial and we will assume it in the following context.
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- This dual problem dates back to Dybvig (1988) and is revived in Jin and Zhou (2008)
We only need to consider the terminal payoff in the form of $F^{-1}(Z)$ where $Z = 1 - F_\rho(\rho)$
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Rewrite the budget constraint

\[
x \geq E \left[ \rho F^{-1}(Z) \right] \\
= E \left[ F_{\rho}^{-1}(1 - Z)F^{-1}(Z) \right] \quad (F_{\rho}^{-1}(F_{\rho}(\rho)) = \rho, \ a.s.) \\
= \int_{0}^{1} F_{\rho}^{-1}(1 - z)F^{-1}(z)dz
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Let $$G(\cdot) := F^{-1}(\cdot)$$

$$\max_{G(\cdot)} U(G(\cdot)) := \int_0^1 u(G(z)) T'(1 - z) dz$$

Subject to

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& \quad \int_0^1 F^{-1}_\rho(1 - z) G(z) dz \leq x_0
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We call it \textit{quantile formulation}
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The optimal solution to the portfolio selection problem must be anti-comonotonic w.r.t the pricing kernel $\rho$. 
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Solvable by Lagrange.
Quantile formulation (Cont’d)

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- Solvable by Lagrange.

- All the aforementioned examples can be solved explicitly.
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We turn the problem into an equivalent optimisation problem — quantile formulation where quantiles serve as the decision variable.

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Incomplete market case can also be dealt with and mutual fund theorem is derived.
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- The quantile formulation can be applied to all the aforementioned models.
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- The quantile formulation can be applied to all the aforementioned models.
- Prospect theory has been solved in Jin and Zhou (2008).
- SP/A model and model with law-invariant coherent risk measure have been solved by He and Zhou recently.