Pricing and Hedging of Credit Derivatives via Nonlinear Filtering

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based on work with
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Overview

1. Introduction: credit risk under incomplete information

2. Pricing and hedging credit derivatives via nonlinear filtering: the [Frey et al., 2007] model. Main ideas:
   - We model evolution of investors beliefs about credit quality, as those are driving credit spreads.
   - We use innovations approach to nonlinear filtering for deriving dynamics of traded credit derivatives.
Attainable correlations for two lognormal variables, $X_1 \sim \text{Ln}(0, 1)$, $X_2 \sim \text{Ln}(0, \sigma^2)$; (from McNeil, Frey, Embrechts, Quantitative Risk Management, Princeton University Press 2005)
1. Credit Risk and Incomplete Information

Basically we have two classes of dynamic credit risk models.

- **Structural models:** Default occurs if the asset value $V_i$ of firm $i$ falls below some threshold $K_i$, interpreted as liability, so that default time is

  $$\tau_i := \inf\{t \geq 0 : V_{t,i} \leq K_i\}.$$ 

  $\tau_i$ is (typically) predictable; dependence between defaults via dependence of the $V_i$.

- **Reduced form models:** Default occurs at the first jump of some point process, typically with stochastic intensity $\lambda_{t,i}$. ($\tau_i$ is totally inaccessible.)

  Usually $\lambda_{t,i} = \lambda_i(t, X_t)$, where $X$ is a common state variable process introducing dependence between default times.
Incomplete information

In both model-classes it makes sense to assume that investors have only limited information about state variables of the model

- **Asset value** $V_i$ is hard to observe precisely $\Rightarrow$ consider firm-value models with noisy information about $V$ (see for instance [Duffie and Lando, 2001], [Jarrow and Protter, 2004], [Coculescu et al., 2006] or [Frey and Schmidt, 2006]).

- In **reduced-form models** state variable process $X$ is usually not associated with observable economic quantities and needs to be backed out from observables such as prices.
Implications of incomplete information

- Under incomplete information \( \tau_i \) typically admits an intensity.

- Natural **two-step-procedure** for pricing: prices are first computed under full information (using Markov property) and then projected on the investor filtration \( \Rightarrow \) Pricing and model calibration naturally lead to **nonlinear filtering problems**.

- **Information-driven default contagion.** In real markets one frequently observes contagion effects, i.e. spreads of non-defaulted firms jump(upward) in reaction to default events. Models with incomplete information mimic this effect: given that firm \( i \) defaults, conditional distribution of the state-variable is updated, \( \Rightarrow \) default intensity of surviving firms increases ([Schönbucher, 2004], [Collin-Dufresne et al., 2003], . . . .)
Some literature (mainly reduced-form models)

- Simple doubly-stochastic models with incomplete information such as [Schönbucher, 2004], [Duffie et al., 2006], extensions in recent work by Giesecke.

- [Frey and Runggaldier, 2007]. Relation between credit risk and nonlinear filtering and analysis of filtering problems in very general reduced-form model; dynamics of credit risky securities not studied.


- [Frey and Runggaldier, 2008] A general overview over nonlinear filtering in term-structure and credit risk models.
2. Our information-based model

Overview. Three layers of information:

1. Underlying default model (full information) Default times $\tau_i$ are conditionally independent doubly-stochastic random times; intensities are driven by a finite-state Markov chain $X$.

2. Market information. Prices of traded credit derivatives are determined by informed market-participants who observe default history and some (abstract) process $Z$ giving $X$ in additive Gaussian noise (market information $\mathcal{F}^M := \mathcal{F}^Y \vee \mathcal{F}^Z$); Filtering results wrt $\mathcal{F}^M$ are used to obtain asset price dynamics.

3. Investor information. $Z$ represents abstract form of ‘insider information’ and is not directly observable. $\Rightarrow$ study pricing and hedging of credit derivatives for secondary-market investors with investor information $\mathcal{F}^I \subset \mathcal{F}^M$. 
Advantages

• Prices are weighted averages of full-information values (the theoretical price wrt $\mathbb{F}^X \lor \mathbb{F}^Y$), so that most computations are done in the underlying Markov model. Since the latter has a simple structure, computations become relatively easy.

• Rich credit-spread dynamics with spread risk (spreads fluctuate in response to fluctuations in $Z$) and default contagion (as defaults lead to an update of the conditional distribution of $X_t$ given $\mathcal{F}_t^M$).

• Model has a natural factor structure with factors given by the conditional probabilities $\pi_t^k = Q(X_t = k \mid \mathbb{F}^M), 1 \leq k \leq K$.

• Great flexibility for calibration. In particular, we may view observed prices as noisy observation of the state $X_t$ and apply calibration via filtering.
Notation

• We work on probability space $(\Omega, \mathcal{F}, Q)$, $Q$ the risk-neutral measure, with filtration $\mathbb{F}$. All processes will be $\mathbb{F}$ adapted.

• We consider portfolio of $m$ firms with default state $Y_t = (Y_{t,1}, \ldots, Y_{t,m})$ for $Y_{t,i} = 1\{\tau_i \leq t\}$. $Y_t^i$ is obtained from $Y_t$ by flipping $i$th coordinate.

Ordered default times denoted by $T_0 < T_1 < \ldots < T_m$; $\xi_n \in \{1, \ldots, m\}$ gives identity of the firm defaulting at $T_n$.

• Default-free interest rate $r(t)$, $t \geq 0$, deterministic. Here $r(t) \equiv 0$. 
The underlying full-information model

Consider a finite-state Markov chain $X$ with $S^X := \{1, \ldots, K\}$ and generator $Q^X$.

A1 The default times are conditionally independent, doubly stochastic random times with $(Q, \mathcal{F})$-default intensity $(\lambda_i(X_t))$.

Implications.

• The processes $Y_{t,j} = \int_0^{t \wedge \tau_j} \lambda_j(X_{s^-}) \, ds$, $1 \leq j \leq m$, are $\mathcal{F}$-martingales.

• $\tau_1, \ldots, \tau_m$ are conditionally independent given $\mathcal{F}_\infty^X$; in particular no joint defaults.

• The pair process $(X, Y)$ is Markov wrt $\mathcal{F}$.
Examples

1. **Homogeneous model** (default intensities of all firms are identical). Default intensities are modelled by some increasing function \( \lambda : \{1, \ldots, K\} \rightarrow (0, \infty) \) of the states of the economy. Elements of \( S^X \) thus represent different states of the economy (1 is the best state and \( K \) the worst state). Various possibilities for generator \( Q^X \); a very simple model takes \( X \) to be constant (Bayesian analysis instead of filtering).

2. **Global- and industry factors.** Assume that we have \( \bar{r} \) different industry groups. Let \( S^X = \{1, \ldots, \kappa\} \times \{0, 1\}^\bar{r} \); write \( X^0, \ldots, X^{\bar{r}} \) for the components of \( X \), modelled as independent Markov chains. \( X^r \) is the state of industry \( r \) which is good \( (X^r = 0) \) or bad \( (X^r = 1) \); \( X^0 \) represents the global factor. Default intensity of firm \( i \) from industry group \( r \) takes the form \( \lambda_i(x) = g_i(x^0) + f_i(x^r) \) for increasing functions \( f_i \) and \( g_i \).
Define the full-information value of a $\mathcal{F}_T^\mathcal{Y}$-measurable claim $H$ (a typical credit derivative) by
\[
\mathbb{E}^Q\left(H \mid \mathcal{F}_t\right) =: h(t, X_t, Y_t);
\] (1)
the last definition makes sense since $(X, Y)$ is Markov w.r.t. $\mathbb{F}$.

Computation of full-information values. Many possibilities:

- Bond prices or legs of a CDS can be computed via Feynman-Kac
- For portfolio products such as CDOs we can use conditional independence and compute Laplace transform of portfolio loss, (as in [Graziano and Rogers, 2006]) or use Poisson- and normal approximations, combined with Monte Carlo.
Market information

Recall that the informational advantage of informed market participants is modelled via observations of a process $Z$. Formally,

**A2** $\mathbb{F}^M = \mathbb{F}^Y \lor \mathbb{F}^Z$, where the $l$-dim. process $Z$ solves the SDE

$$dZ_t = a(X_t)dt + dB_t.$$ 

Here, $B$ is an $l$-dim standard $\mathbb{F}$-Brownian motion independent of $X$ and $Y$, and $a(\cdot)$ is a function from $S^X$ to $\mathbb{R}^l$.

**Notation.** Given a generic RCLL process $U$, we denote by $\hat{U}$ the optional projection of $U$ w.r.t. the market filtration $\mathbb{F}^M$; recall that $\hat{U}$ is a right continuous process with $\hat{U}_t = \mathbb{E}(U_t|\mathcal{F}^M_t)$ for all $t \geq 0$. 
3. Dynamics of Security Prices

**Traded securities.** We consider $N$ liquidly traded credit derivatives (eg. corporate bonds) with maturity $T$ and $\mathcal{F}_T^Y$-measurable payoff $P_{T,1}, \ldots, P_{T,N}$. We use martingale modelling:

**A3** Prices of traded securities are given by $\hat{p}_{t,i} := E^Q(P_{T,i}\mid \mathcal{F}_t^M)$.

**Market-pricing.** Denote by $p_i(t, X_t, Y_t)$ the full-information value of security $i$. We get from iterated conditional expectations

$$\hat{p}_{t,i} = E(E(P_{T,i}\mid \mathcal{F}_t) \mid \mathcal{F}_t^M) = E(p_i(t, X_t, Y_t)\mid \mathcal{F}_t^M).$$

(2)

Note that this is solved if we know the conditional distribution of $X_t$ given $\mathcal{F}_t^M$ (a nonlinear filtering problem).

**Goal.** Study the dynamics of traded security prices $\hat{p}_{t,i}$; this is a prerequisite for hedging and risk management.
Innovations processes

As a first towards determining the dynamics of the traded security prices step we introduce the innovations processes:

\[ M_{t,j} := Y_{t,j} - \int_0^{t \wedge \tau_j} \lambda_j(X_s^-) \, ds , \quad j = 1, \ldots , m \]

\[ \mu_{t,i} := Z_{t,i} - \int_0^{t} a_i(X_s) \, ds , \quad i = 1, \ldots , l. \]

**Properties.**

- \( M_j \) is an \( \mathbb{F}^M \)-martingale and \( \mu \) is \( \mathbb{F}^M \)-Brownian motion.
- Every \( \mathbb{F}^M \)-martingale can be represented as stochastic integral wrt \( M \) and \( \mu \).
General filtering equations

Proposition 1 (General filtering equations). Consider a $\mathbb{F}$-semimartingale of the form $J_t = J_0 + \int_0^t A_s ds + M^J_t$, $M^J$ an $\mathbb{F}$-martingale with $[M^J, B] = 0$. Suppose that $[J, Y_i]_t = \int_0^t R^{J,i}_s dY_{s,i}$. Then $\hat{J}$ has the representation

$$\hat{J}_t = \hat{J}_0 + \int_0^t \hat{A}_s ds + \int_0^t \gamma_s^\top dM_s + \int_0^t \alpha_s^\top d\mu_s; \quad (3)$$

$\gamma$ and $\alpha$ are given by

$$\alpha_t = \widehat{J}_t a(X_t) - \hat{J}_t a(X_t), \quad (4)$$

$$\gamma_{t,i} = \frac{1}{(\hat{\lambda}_i)_{t-}} \left( (\hat{J}\lambda_i)_{t-} + \hat{J}_{t-}(\hat{\lambda}_i)_{t-} + (R^{J,i}\lambda_i)_{t-} \right). \quad (5)$$

Proof based on innovations approach to nonlinear filtering.
Theorem 2. Under A1 - A3 the (discounted) price process of the traded securities has the martingale representation

\[ \hat{p}_{t,i} = \hat{p}_{0,i} + \int_0^t \gamma_{s}^\top \, dM_s + \int_0^t \alpha_{s}^\top \, d\mu_s, \text{ with} \]

\[ \alpha_t^{\hat{p}_i} = \hat{p}_{t,i} \cdot a_t - \hat{p}_{t,i} \hat{a}_t \]

\[ \gamma_{t,j}^{\hat{p}_i} = \text{as in (5) with } R_{t}^{p_i,j} = p_i(t, X_t, Y_t^i) - p(t, X_t, Y_t). \]

The predictable quadratic variations of the asset prices with respect to the market information \( \mathbb{F}^M \) satisfy

\[ d\langle \hat{p}_i, \hat{p}_j \rangle^M_t = v^{ij}_t \, dt \text{ with} \]

\[ v^{ij}_t = \sum_{n=1}^{m} \gamma_{t,n}^{\hat{p}_i} \gamma_{t,n}^{\hat{p}_j} \hat{\lambda}_{t-,n} + \sum_{n=1}^{l} \alpha_{t-,n}^{\hat{p}_i} \alpha_{t-,n}^{\hat{p}_j}. \]
Filtering

Define the conditional probability vector \( \mathbf{\pi}_t = (\pi^1_t, \ldots, \pi^K_t)^\top \) with \( \pi^k_t := Q(X_t = k | \mathcal{F}_t^M) \). \( \mathbf{\pi}_t \) is the natural state variable; under market information \( \mathbb{F}^M \) all quantities of interest are functions of \( \mathbf{\pi}_t \).

Kushner-Stratonovich equation. \((K\text{-dim SDE-system for } \mathbf{\pi})\) Let \( q(\iota, k), 1 \leq \iota, k \leq K \) denote generator matrix of \( X \). Then

\[
d\pi^k_t = \sum_{\iota=1}^{K} q(\iota, k) \pi^\iota_t dt + (\gamma^k(\mathbf{\pi}_t^-)) \top dM_t + (\alpha^k(\mathbf{\pi}_t)) \top d\mu_t, \quad \text{with}
\]

\[
\gamma^k_j(\mathbf{\pi}) = \pi^k \left( \frac{\lambda_j(k)}{\sum_{n=1}^{K} \lambda_j(n) \pi_n} - 1 \right), \quad 1 \leq j \leq m, \quad (8)
\]

\[
\alpha^k(\mathbf{\pi}) = \pi^k \left( a(k) - \sum_{n=1}^{K} \pi_n a(n) \right). \quad (9)
\]
Default contagion

- Updating at the default time $\tau_j$.

$$\Delta \pi^k_{\tau_j} = \pi^k_{\tau_j-} \left( \frac{\lambda_j(k)}{\sum_{n=1}^{K} \lambda_j(n) \pi^n_{\tau_j-}} - 1 \right).$$

- Default contagion. At $\tau_j$ default intensity of firm $i$ jumps:

$$\hat{\lambda}_{\tau_j,i} - \hat{\lambda}_{\tau_j-,i} = \sum_{k=1}^{K} \lambda_i(k) \cdot \pi^k_{\tau_j-} \left( \frac{\lambda_j(k)}{\sum_{l=1}^{K} \lambda_j(l) \pi^l_{\tau_j-}} - 1 \right) = \frac{\text{cov}_{\pi^\tau_{\tau_j-}} (\lambda_i, \lambda_j)}{\mathbb{E}_{\pi^\tau_{\tau_j-}} (\lambda_j)}.$$
The filter in action

![Graph](image1)

![Graph](image2)
4. Secondary market investors

Recall that secondary market investors do not observe \( Z \). Their information set is given by \( \mathbb{F}^I \subseteq \mathbb{F}^M \); typically \( \mathbb{F}^I \) contains default history and noisy price information.

**Pricing.** Consider non-traded \( \mathcal{F}^Y_T \)-measurable claim \( H \). Define its secondary-market value as \( \mathbb{E}(H|\mathcal{F}_t^I) \). Let \( h_t(X_t) = E(H | \mathcal{F}_t) \) (full-information value of \( H \)). We get from iterated conditional expectations

\[
\mathbb{E}(H|\mathcal{F}_t^I) = \mathbb{E}(\mathbb{E}(H|\mathcal{F}_t^M)|\mathcal{F}_t^I) = \sum_{k=1}^{K} E(\pi^k_t | \mathcal{F}_t^I) h_t(k),
\]

i.e. pricing wrt \( \mathbb{F}^I \) reduces to finding \( E(\pi^k_t | \mathcal{F}_t^I) \).
Hedging. We look for risk-minimizing strategies under restricted information in the sense of [Schweizer, 1994].

- Quadratic criterion combines well with incomplete information
- On credit markets it is natural to minimize risk wrt martingale measure $Q$ as historical default intensities are hard to determine.

The risk-minimizing strategy $\theta^H$ can be computed by suitably projecting the $\mathbb{F}^M$-risk-minimizing hedging strategy $\xi^H_t$ on the set of $\mathbb{F}^I$-predictable strategies. For instance we get with only one traded asset that $\theta_t$ is left-continuous version of

$$E(v_t \xi^H_t \mid \mathcal{F}^I_t) / E(v_t \mid \mathcal{F}^I_t).$$

Recall that $v_t$ and $\xi_t$ are nonlinear functions of $\pi_t$. $\Rightarrow$ We need to determine $\nu_t(d\pi)$, the conditional distribution of $\pi$ given $\mathcal{F}^I_t$. 


Modelling $F^I$ and Calibration Strategies

Pragmatic calibration. Here prices of traded securities are observable. Recall that $\hat{p}_{t,i} = \sum_{k=1}^{K} \pi^k_t p_i(t, k, Y_t)$. If $N \geq K$ (more securities than states) and if the matrix $p(t, Y_t) := (p_i(t, k, Y_t))$ of fundamental values has full rank, the vector $\pi_t$ could be implied by standard calibration:

$$\pi_t = \arg\min_{\{\pi \geq 0, \sum_{k=1}^{K} \pi_k = 1\}} \frac{1}{N} \sum_{n=1}^{N} w_n \left( \hat{p}_{n} - \sum_{k=1}^{K} p_n(t, k, Y_t) \pi_k \right)^2,$$

for suitable weights $w_1, \ldots, w_N$.

In that case pricing and hedging for secondary market investors and informed market participants coincides.
Left: itraxx spreads from last winter for different maturities; Right: homogeneous model with 3 states and state probabilities calibrated to itraxx; note that probability of worst state increases over time.
Calibration via filtering

Alternatively, assume that $\mathbb{F}^I = \mathbb{F}^Y \vee \mathbb{F}^U$ where the $N$-dim process $U$ solves the SDE

$$dU_t = \hat{p}_t dt + dW_t = p(t, Y_t) \pi_t dt + dW_t$$

for a Brownian motion $W$ independent of $X, Y, Z$. $U$ can be viewed as cumulative noisy price information of the traded assets $\hat{p}_1, \ldots, \hat{p}_N$; noise reflects observation errors and model errors.

Recall that $\pi$ solves the KS-equation (7). Hence computation of the conditional distribution of $\pi_t$ given $\mathcal{F}_t^I$ is a nonlinear filtering problem with signal process $\pi$ and observation process $U$ and $Y$. 
Filtering problem for secondary-market investors

Challenging problem:

- Observations of mixed type; Joint jumps of state process $\pi$ and observation $Y$ at defaults (see for instance [Frey and Runggaldier, 2007])
- Typically high-dimensional problem $\Rightarrow$ use particle filtering as in [Crisan and Lyons, 1999]
- Numerical analysis work in progress.
References


