Nonlinear Optimization

Spring 2008 Princeton University Prof. Eckstein

Homework 2: Convex Analysis Foundations

Due Wednesday, February 19

This assignment will use the same notation as the textbook: $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

1. Suppose $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is convex, A is an $m \times n$ real matrix, and $b \in \mathbb{R}^m$. Show that the function $g(x) = f(Ax + b)$ is convex.

2. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *quasiconvex* if its level sets $M_\beta = \{x \in \mathbb{R}^n \mid f(x) \leq \beta\}$ are convex for all $\beta \in \mathbb{R}$.

   (a) Show that f is quasiconvex if and only if, for all $x^1, x^2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, one has $f(\lambda x^1 + (1 - \lambda)x^2) \leq \max\{f(x^1), f(x^2)\}$.

   (b) Give a simple example of a quasiconvex function on $\mathbb{R}$ that is not convex.

   (c) Suppose that $X \subseteq \mathbb{R}^n$ is a convex set and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is quasiconvex. Prove that if x^* is a *strict* local minimum of f on X, then it must be the global minimum.

   (d) Give an example of a quasiconvex function on $\mathbb{R}$ that has a non-strict local minimum that is not global.

3. This problem will lead you through the proof of a fact used in class: if the set $C \subseteq \mathbb{R}^n$ is convex, then $\operatorname{int} \operatorname{cl} C = \operatorname{int} C$. Below, assume C is convex, and $\operatorname{int} C \neq \emptyset$.

   (a) *The line segment principle I:* Suppose $x \in \operatorname{int} C$ and $y \in C$. Show that for all $\lambda \in (0, 1]$, $\lambda x + (1 - \lambda)y \in \operatorname{int} C$. Hint: if $\epsilon > 0$ is such that $B(x, \epsilon) \subseteq C$, use convexity to show that $B(\lambda x + (1 - \lambda)y, \lambda\epsilon) \subseteq C$.

   (b) *The line segment principle II:* Now make the weaker assumption that $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$. Show that it is still true that $\lambda x + (1 - \lambda)y \in \operatorname{int} C$ for all $\lambda \in (0, 1]$. Hint: take ϵ as in question 3a and $\{y^k\} \subseteq C$ converging to y; for each given value of λ, show that for k sufficiently large, $\lambda x + (1 - \lambda)y$ falls within $B(\lambda x + (1 - \lambda)y^k, \lambda\epsilon)$, which in turn lies with C.

   (c) Suppose $x \in \operatorname{int} C$ and $z \in \operatorname{int} \operatorname{cl} C$. Show that there must exist points $y \in \operatorname{cl} C$ such that z is a convex combination of x and y.

   (d) Assume $\operatorname{int} C \neq \emptyset$. Combine questions 3b and 3c to argue that any $z \in \operatorname{int} \operatorname{cl} C$ is also in $\operatorname{int} C$. Finally, argue that this fact proves $\operatorname{int} \operatorname{cl} C = \operatorname{int} C$.

   Note: strictly speaking, this exercise has only proven $\operatorname{int} \operatorname{cl} C = \operatorname{int} C$ when $\operatorname{int} C \neq \emptyset$. The result also holds when $\operatorname{int} C = \emptyset$, as can be seen from Lemma 2.9 in the textbook.

4. Give an example of a (necessarily nonconvex) set X for which $\operatorname{int} \operatorname{cl} X \neq \operatorname{int} X$.