

SPDE and portfolio choice

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Performance measurement of investment strategies



Market environment

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$; $W = (W^1, \dots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \leq i \leq k \quad \begin{cases} dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t \right) , & S_0^i > 0 \\ dB_t = r_t B_t dt , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}, \sigma_t^i \in \mathbb{R}^d$ bounded and \mathcal{F}_t -measurable stochastic processes

- Postulate existence of an \mathcal{F}_t -measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

Market environment

- Self-financing investment strategies $\pi_t^0, \pi_t = (\pi_t^1, \dots, \pi_t^i, \dots, \pi_t^k)$
- Present value of this allocation

$$X_t = \sum_{i=0}^k \pi_t^i$$

$$dX_t = \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot (\lambda_t dt + dW_t)$$

$$= \sigma_t \pi_t \cdot (\lambda_t dt + dW_t)$$

Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the **end** of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

Backwards in time generation of the indirect utility

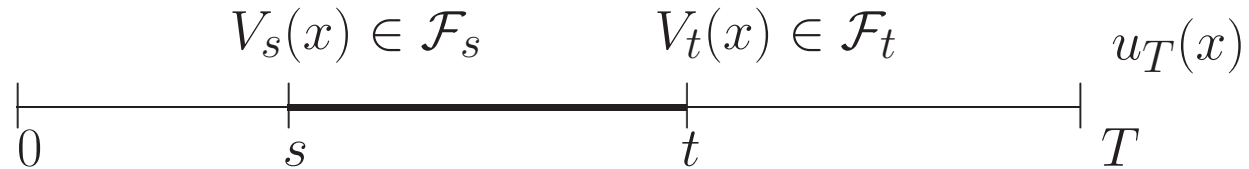
$$V_s(x) = \sup_{\pi} E_{\mathbb{P}}(u_T(X_T^{\pi}) | \mathcal{F}_s; X_s^{\pi} = x)$$

$$V_s(x) = \sup_{\pi} E_{\mathbb{P}}(V_t(X_t^{\pi}) | \mathcal{F}_s; X_s^{\pi} = x) \quad (\text{DPP})$$

$$V_s(x) = E_{\mathbb{P}}(V_t(X_t^{\pi^*}) | \mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility
for all $t \in [0, T)$

Investment performance process



- For each self-financing strategy, represented by π , the associated wealth X_t^π satisfies

$$E_{\mathbb{P}}(V_t(X_t^\pi) | \mathcal{F}_s) \leq V_s(X_s^\pi), \quad 0 \leq s \leq t \leq T$$

- There exists a self-financing strategy, represented by π^* , for which the associated wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(V_t(X_t^{\pi^*}) | \mathcal{F}_s) = V_s(X_s^{\pi^*}), \quad 0 \leq s \leq t \leq T$$

Investment performance process

$$V_{t,T}(x) \in \mathcal{F}_t, \quad 0 \leq t \leq T$$

- $V_{t,T}(X_t^\pi)$ is a supermartingale
- $V_{t,T}(X_t^{\pi^*})$ is a martingale
- $V_{t,T}(x)$ is the terminal utility in trading subintervals $[s, t]$, $0 \leq s \leq t$

Observations

- $V_{T,T}(x)$ is chosen exogeneously to the market
- Choice of horizon possibly restrictive
- More realistic to have random terminal data, $V_{T,T}(x, \omega) = U(x, \omega)$

Investment performance process

$U_t(x)$ is an \mathcal{F}_t -adapted process, $t \geq 0$

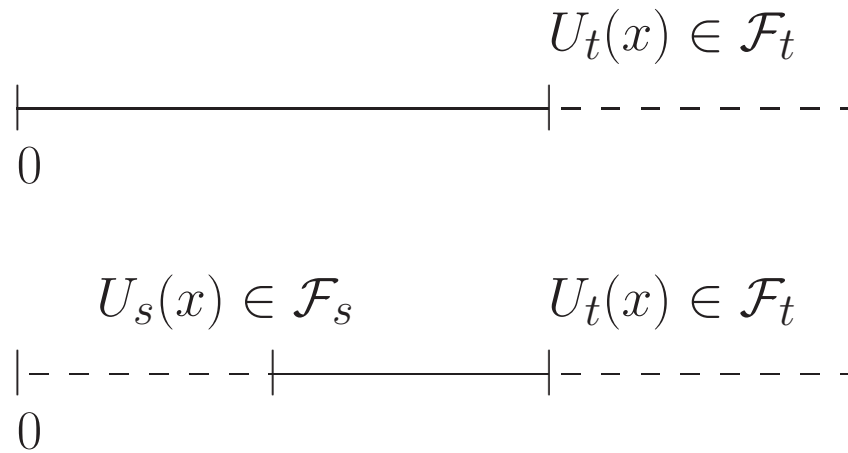
- The mapping $x \rightarrow U_t(x)$ is increasing and concave
- For each self-financing strategy, represented by π , the associated (discounted) wealth X_t^π satisfies

$$E_{\mathbb{P}}(U_t(X_t^\pi) \mid \mathcal{F}_s) \leq U_s(X_s^\pi), \quad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^* , for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U_t(X_t^{\pi^*}) \mid \mathcal{F}_s) = U_s(X_s^{\pi^*}), \quad 0 \leq s \leq t$$

Optimality across times



$$U_s(x) = \sup_{\mathcal{A}} E(U_t(X_t^\pi) | \mathcal{F}_s, X_s = x)$$

- Does such a process always exist?
- Is it unique?

Forward performance process

A datum $u_0(x)$ is assigned at the **beginning** of
the trading horizon, $t = 0$

$$U_0(x) = u_0(x)$$

Forward in time generation of optimal performance

$$E_{\mathbb{P}}(U_t(X_t^{\pi}) | \mathcal{F}_s) \leq U_s(X_s^{\pi}), \quad 0 \leq s \leq t$$

$$E_{\mathbb{P}}(U_t(X_t^{\pi^*}) | \mathcal{F}_s) = U_s(X_s^{\pi^*}), \quad 0 \leq s \leq t$$

Many difficulties due to “inverse in time”

nature of the problem

The stochastic PDE of the forward performance process



The forward performance SPDE

Let $U(x, t)$ be an \mathcal{F}_t -measurable process such that the mapping $x \rightarrow U(x, t)$ is increasing and concave. Let also $U = U(x, t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(U\lambda + a)|^2}{\mathcal{A}^2 U} dt + a \cdot dW$$

where $a = a(x, t)$ is an \mathcal{F}_t -adapted process, while $\mathcal{A} = \frac{\partial}{\partial x}$.

Then $U(x, t)$ is a forward performance process.

The process a may depend on t, x, U , its spatial derivatives etc.

At the optimum

- The optimal portfolio vector π^* is given in the feedback form

$$\pi_t^* = \pi^*(X_t^*, t) = -\sigma^+ \frac{\mathcal{A}(U\lambda + a)}{\mathcal{A}^2 U} (X_t^*, t)$$

- The optimal wealth process X^* solves

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}(U\lambda + a)}{\mathcal{A}^2 U} (X_t^*, t) (\lambda dt + dW_t)$$

Intuition for the structure of the forward performance process

- Assume that $U = U(x, t)$ solves

$$dU(x, t) = b(x, t) dt + a(x, t) \cdot dW_t$$

where b, a are \mathcal{F}_t -measurable processes.

- Recall that for an arbitrary admissible portfolio π , the associated wealth process, X^π , solves

$$dX_t^\pi = \sigma_t \pi_t (\lambda_t dt + dW_t)$$

- Apply the Ito-Ventzell formula to $U(X_t^\pi, t)$ we obtain

$$\begin{aligned} dU(X_t^\pi, t) &= b(X_t^\pi, t) dt + a(X_t^\pi, t) \cdot dW_t \\ &+ U_x(X_t^\pi, t) dX_t^\pi + \frac{1}{2} U_{xx}(X_t^\pi, t) d\langle X^\pi \rangle_t + a_x(X_t^\pi, t) d\langle W, X^\pi \rangle_t \\ &= \left(b(X_t^\pi, t) + U_x(X_t^\pi, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x(X_t^\pi, t) + \frac{1}{2} U_{xx}(X_t^\pi, t) |\sigma_t \pi_t|^2 \right) dt \\ &\quad + (a(X_t^\pi, t) + U_x(X_t^\pi, t) \sigma_t \pi_t) \cdot dW_t \end{aligned}$$

Intuition (continued)

- By the monotonicity and concavity assumptions, the quantity

$$\sup_{\pi} \left(U_x (X_t^{\pi}, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x (X_t^{\pi}, t) + \frac{1}{2} U_{xx} (X_t^{\pi}, t) |\sigma_t \pi_t|^2 \right)$$

is well defined.

- Calculating the optimum π^* yields

$$\pi_t^* = -\sigma_t^+ \frac{U_x (X_t^{\pi^*}, t) \lambda_t + a_x (X_t^{\pi^*}, t)}{U_{xx} (X_t^{\pi^*}, t)}$$

- Deduce that the above supremum is given by

$$M^* (X_t^{\pi^*}, t) = - \frac{|\sigma_t \sigma_t^+ (U_x (X_t^{\pi^*}, t) \lambda_t + a_x (X_t^{\pi^*}, t))|^2}{2U_{xx} (X_t^{\pi^*}, t)}$$

- Choose the drift coefficient

$$b(x, t) = -M^*(x, t)$$

Solutions to the forward performance SPDE

$$dU = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(U\lambda + a)|^2}{\mathcal{A}^2 U} dt + a \cdot dW$$

Local differential coefficients

$$a(x, t) = F(x, t, U(x, t), U_x(x, t))$$

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on U_x and U_{xx}
- The equation is not (degenerate) elliptic

Choices of volatility coefficient

- The deterministic case: $a(x, t) = 0$

The forward performance SPDE simplifies to

$$dU = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}(U\lambda)|^2}{\mathcal{A}^2 U} dt$$

The process

$$U(x, t) = u(x, A_t) \quad \text{with} \quad A_t = \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds$$

with $u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$, increasing and concave with respect to x , and solving

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

is a solution.

MZ (2006)

Berrier, Rogers and Tehranchi (2007)

- $a(x, t) = 0$

σ, λ constants and u separable (in space and time)

The forward performance process reduces to a deterministic function.

$$U(x, t) = u(x, t)$$

$$u(x, t) = -e^{-x + \frac{t}{2}} \quad \text{or} \quad u(x, t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)} \lambda^2 t}$$

Horizon-unbiased utilities

Henderson-Hobson (2006)

- $a(x, t) = k, \quad k \in \mathbb{R}$

$$U(x, t) = u(x, A_t) + kW_t$$

The “market-view” case

$$a = U\phi, \quad \phi \text{ is a } d\text{-dim } \mathcal{F}_t\text{-adapted process}$$

- The forward performance SPDE becomes

$$dU = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}U (\lambda + \phi)|^2}{\mathcal{A}^2 U} dt + U\phi \cdot dW$$

- Define the processes Z and A by

$$dZ = Z\phi \cdot dW \quad \text{and} \quad Z_0 = 1$$

and

$$A_t = \int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s)|^2 ds$$

- The process $U = U(x, t)$

$$U(x, t) = u(x, A_t) Z_t$$

with u solving

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

is a solution

The “benchmark” case

$a(x, t) = -xU(x, t)\delta$, δ is a d -dim \mathcal{F}_t -adapted process

- The forward performance SPDE becomes

$$dU(x, t) = \frac{1}{2} \frac{\left| \sigma_t \sigma_t^+ (U_x(x, t) (\lambda_t - \delta_t) - x U_{xx}(x, t)) \right|^2}{U_{xx}(x, t)} dt - x U_x(x, t) \delta_t \cdot dW_t$$

- Define the processes Y and A by

$$dY_t = Y_t \delta_t (\lambda_t dt + dW_t) \quad \text{with} \quad Y_0 = 1$$

and

$$A_t = \int_0^t \left| \sigma_s \sigma_s^+ \lambda_s - \delta_s \right|^2 ds.$$

- Assume $\sigma \sigma^+ \delta = \delta$
- The process

$$U = U(x, t) = u\left(\frac{x}{Y_t}, A_t\right)$$

with u as before is a forward performance.

A general case

$$a(x, t) = -xU_x(x, t)\delta + U(x, t)\phi$$

- The forward performance SPDE becomes

$$dU(x, t) = \frac{1}{2} \frac{\left| \sigma_t \sigma_t^+ (U_x(x, t) ((\lambda_t + \phi_t) - \delta_t) - xU_{xx}(x, t)\delta_t) \right|^2}{U_{xx}(x)} dt \\ + (-xU_x(x, t)\delta_t + U(x, t)\phi_t) \cdot dW_t$$

- Recall the "benchmark" and "market view processes"

$$dY_t = Y_t \delta_t (\lambda_t dt + dW_t) \quad \text{with} \quad Y = 1$$

and

$$dZ_t = Z_t \phi_t \cdot dW_t \quad \text{with} \quad Z = 1$$

- Define the process

$$A_t = \int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s) - \delta_s|^2 ds$$

- The process

$$U = U(x, t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

is a forward performance

MZ (2006, 2007)

The u-pde

An important differential object is the fully non-linear pde

$$u_t u_{xx} = \frac{1}{2} u_x^2 \quad t > 0,$$

with $u_0(x) = U(x, 0)$.

The local risk tolerance

A quantity that enters in the explicit representation of the optimal portfolios

$$r = -\frac{u_x}{u_{xx}}$$

Modelling considerations

Three related pdes

- Fast diffusion equation for risk tolerance

$$\begin{cases} r_t + \frac{1}{2}r^2 r_{xx} = 0 \\ r(x, 0) = r_0(x) \end{cases} \quad (\text{FDE})$$

Conductivity : r^2

- The transport equation

$$u_t + \frac{1}{2}ru_x = 0$$

with u_0 such that $r_0 = r(x, 0) = -\frac{u_0'(x)}{u_0''(x)}$

- Porous medium equation for risk aversion $\gamma = r^{-1}$

$$\gamma_t = \frac{1}{2}F(\gamma)_{xx} \quad \text{with} \quad F(\gamma) = \gamma^{-1}$$

Difficulties

- **Differential input equation:** $u_t u_{xx} = \frac{1}{2}u_x^2$

Inverse problem and fully nonlinear

- **Transport equation:** $u_t + \frac{1}{2}ru_x = 0$

Shocks, solutions past singularities

- **Fast diffusion equation:** $r_t + \frac{1}{2}r^2r_{xx} = 0$

Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- **Porous medium equation:** $\gamma_t = \frac{1}{2}\left(\frac{1}{\gamma}\right)_{xx}$

Majority of results for (PME), $\gamma_t = (\gamma^m)_{xx}$, are for $m > 1$, partial results for $-1 < m < 0$

An example of local risk tolerance

(MZ (2006) and Z-Zhou (2007))

$$r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \quad \alpha, \beta > 0$$

(Very) special cases

$$r(x, t; 0, \beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x, t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in R$$

$$r(x, t; 1, 0) = |x| \quad \longrightarrow \quad u(x, t) = \log x - \frac{t}{2}, \quad x > 0$$

$$r(x, t; \alpha, 0) = \sqrt{\alpha} |x| \quad \longrightarrow \quad u(x, t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)} t}, \quad x \geq 0, \quad \gamma = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}$$

Optimal allocations



Optimal portfolio vector

- The SPDE for the forward performance process

$$dU = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}(U\lambda + a)|^2}{\mathcal{A}^2 U} dt + a \cdot dW$$

- The optimal portfolio vector

$$\pi_t^* = \pi^*(t, X_t^*) = -\sigma^+ \frac{\mathcal{A}(U\lambda + a)}{\mathcal{A}^2 U} (X_t^*, t)$$

- The optimal wealth process

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}(U\lambda + a)}{\mathcal{A}^2 U} (X_t^*, t) (\lambda dt + dW_t)$$

Optimal portfolios in the MZ example



The structure of optimal portfolios

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

**Stochastic input
Market**

$$(Y_t, Z_t)$$
$$\lambda_t, \sigma_t, \delta_t, \phi_t$$
$$A_t$$

**Differential input
Individual**

$$\text{wealth } x$$
$$\text{risk tolerance } r(x, t)$$
$$r_t + \frac{1}{2} r^2 r_{xx} = 0$$

$$U(x, t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

$\frac{1}{Y_t} \pi_t^*$ is a *linear* combination
of (benchmarked) optimal wealth
and subordinated (benchmarked) risk tolerance

Optimal asset allocation

- Let X_t^* be the optimal wealth, Y_t the benchmark and A_t the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define

$$\tilde{X}_t^* \triangleq \frac{X_t^*}{Y_t} \quad \text{and} \quad \tilde{R}_t^* \triangleq r(\tilde{X}_t^*, A_t)$$

Optimal (benchmarked) portfolios

$$\hat{\pi}_t^* \triangleq \frac{1}{Y_t} \pi_t^* = m_t \tilde{X}_t^* + n_t \tilde{R}_t^*$$

$$m_t = \sigma_t^+ \delta_t \quad n_t = \sigma_t^+ (\lambda_t + \phi_t - \delta_t)$$

Stochastic evolution of wealth-risk tolerance
Explicit construction of optimal processes



A system of SDEs at the optimum

$$\widetilde{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \widetilde{R}_t^* = r(\widetilde{X}_t^*, A_t)$$

$$\begin{cases} d\widetilde{X}_t^* = r(\widetilde{X}_t^*, A_t)(\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\ d\widetilde{R}_t^* = r_x(\widetilde{X}_t^*, A_t) d\widetilde{X}_t^* \end{cases}$$

The optimal wealth and portfolios are explicitly constructed

if the function $r(x, t)$ is known

Solutions of the fast diffusion risk tolerance pde

$$r_t + \frac{1}{2}r^2 r_{xx} = 0$$

Positive and increasing space-time harmonic functions

- Assume that $h(x, t)$ is positive, increasing in x , and satisfies

$$h_t + \frac{1}{2}h_{xx} = 0$$

- Then, it follows from Widder's theorem, that there exists a finite positive Borel measure such that

$$h(x, t) = \int_0^\infty e^{yx - \frac{1}{2}y^2 t} \nu(dy)$$

Risk tolerance function

- Take a positive and increasing space time harmonic function $h(x, t)$
- Define the risk tolerance function $r(x, t)$ by

$$r(x, t) = h_x(h^{-1}(x, t), t)$$

- Then $r(x, t)$ solves the FDE

$$r_t + \frac{1}{2}r^2 r_{xx} = 0, \quad r(0, t) = 0$$

The differential input function u

- Define the function

$$u(x, t) = \int_0^x \exp\left(-h^{-1}(y, t) + \frac{1}{2}t\right) dy$$

- Then u solves

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

- Alternatively, use $r(x, t) = h_x(h^{-1}(x, t), t)$ and the transport equation

$$u_t + \frac{1}{2} r u_x = 0$$

Example

- Consider the case when the positive Borel measure is a Dirac delta, i.e.,

$$\nu = \delta_\gamma, \quad \gamma > 0$$

- Then

$$h(x, t) = e^{\gamma x - \frac{1}{2}\gamma^2 t},$$

$$h^{-1}(x, t) = \frac{1}{\gamma} \left(\log x + \frac{1}{2}\gamma^2 t \right),$$

$$r(x, t) = \lambda x,$$

$$u(x, t) = \frac{\gamma}{\gamma - 1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{1}{2}(\gamma-1)t}$$

Globally defined solutions to the u -pde and the FDE

- Assume that for a finite positive Borel measure on \mathbb{R}

$$\int_{\mathbb{R}} e^{-yx} \left(1 + |y| + \frac{1}{|y|} \right) \nu(dy) < \infty$$

- Assume that the equation below has a solution

$$b'(t) = - \frac{1}{2} \frac{\int_{\mathbb{R}} e^{-yb(t) - \frac{1}{2}y^2t} \nu(dy)}{\int_{\mathbb{R}} e^{-yb(t) - \frac{1}{2}y^2t} y \nu(dy)}, \quad b(0) = b_0$$

Increasing space-time harmonic functions

- Define the function

$$h(x, t) = \int_{\mathbb{R}} \left(-e^{-yx - \frac{1}{2}y^2t} + e^{-yb(t) - \frac{1}{2}y^2t} \right) \frac{1}{y} v(dy)$$

- The above function satisfies

$$h_t + \frac{1}{2}h_{xx} = 0 ,$$

$$h(b(t), t) = 0 \quad \Longleftrightarrow \quad b(t) = h^{-1}(0, t)$$

Risk tolerance function

- The solution to the fast diffusion risk tolerance pde is given by

$$r(x, t) = h_x(h^{-1}(x, t), t)$$

Example

- For positive constants a and b define

$$h(x, t) = \frac{b}{a} \exp\left(-\frac{1}{2}a^2t\right) \sinh(ax)$$

- Observe that

$$r(x, t) = \sqrt{a^2x^2 + b^2 \exp(-a^2t)}$$

- The corresponding $u(x, t)$ function can be calculated explicitly
- The above class covers the classical exponential, logarithmic and power cases

Notice that $r(x, t)$ is globally defined

Optimal wealth and risk tolerance processes

- Define the process

$$M_t = \int_0^t \sigma \sigma^+ \lambda_s dW_s$$

- Note that

$$A_t = \langle M \rangle_t$$

- Optimal wealth process

$$X_t^{x,*} = h(h^{-1}(x, 0) + A_t + M_t, A_t)$$

- Risk tolerance process

$$R_t^{x,*} = r(X_t^{x,*}, A_t) = h_x(h^{-1}(x, 0) + A_t + M_t, A_t)$$

Construction

- Initial data $u_0(x)$, or $r_0(x)$, yields $h(x, 0)$
- Backward heat equation for h
- Solution $h(x, t)$
- Risk tolerance function $r(x, t) = h_x(h^{-1}(x, t), t)$
- Market input $M_t = \int_0^t \sigma_s \sigma_s^+ \lambda_s dW_s$

Construction

- Optimal wealth $X_t^{x,*} = h \left(h^{-1}(x, 0) + \langle M \rangle_t + M_t, \langle M \rangle_t \right)$

- Optimal risk tolerance $r(X_t^{x,*}, t) = h_x \left(h^{-1}(X_t^{x,*}, t), t \right)$

- Optimal portfolio

$$\pi_t^* = k_t^1 X_t^{x,*} + k_t^2 r(X_t^{x,*}, t)$$

- Distributional properties of optimal wealth

- Specification of initial data $h(x, 0)$?

- Inference of initial data from the investor's wish list.