

Twists and Turns in the Skew and Term Structure:
Calibration 2.0 of the Implied Volatility Surface

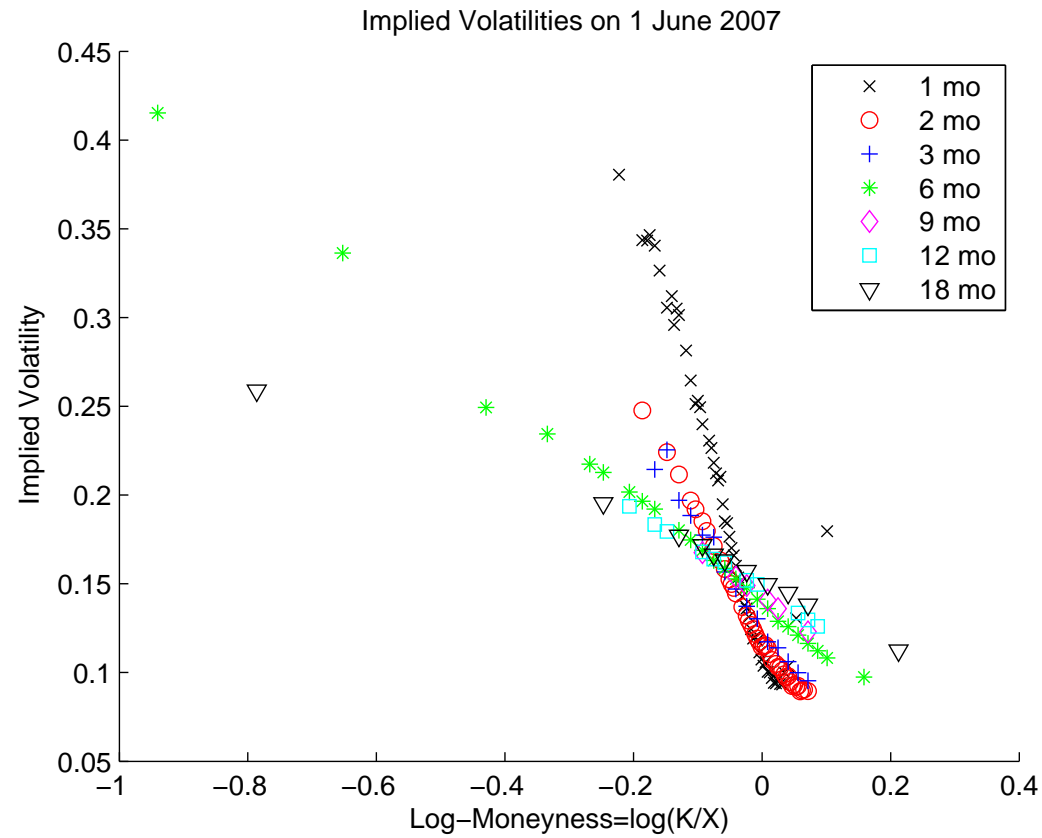
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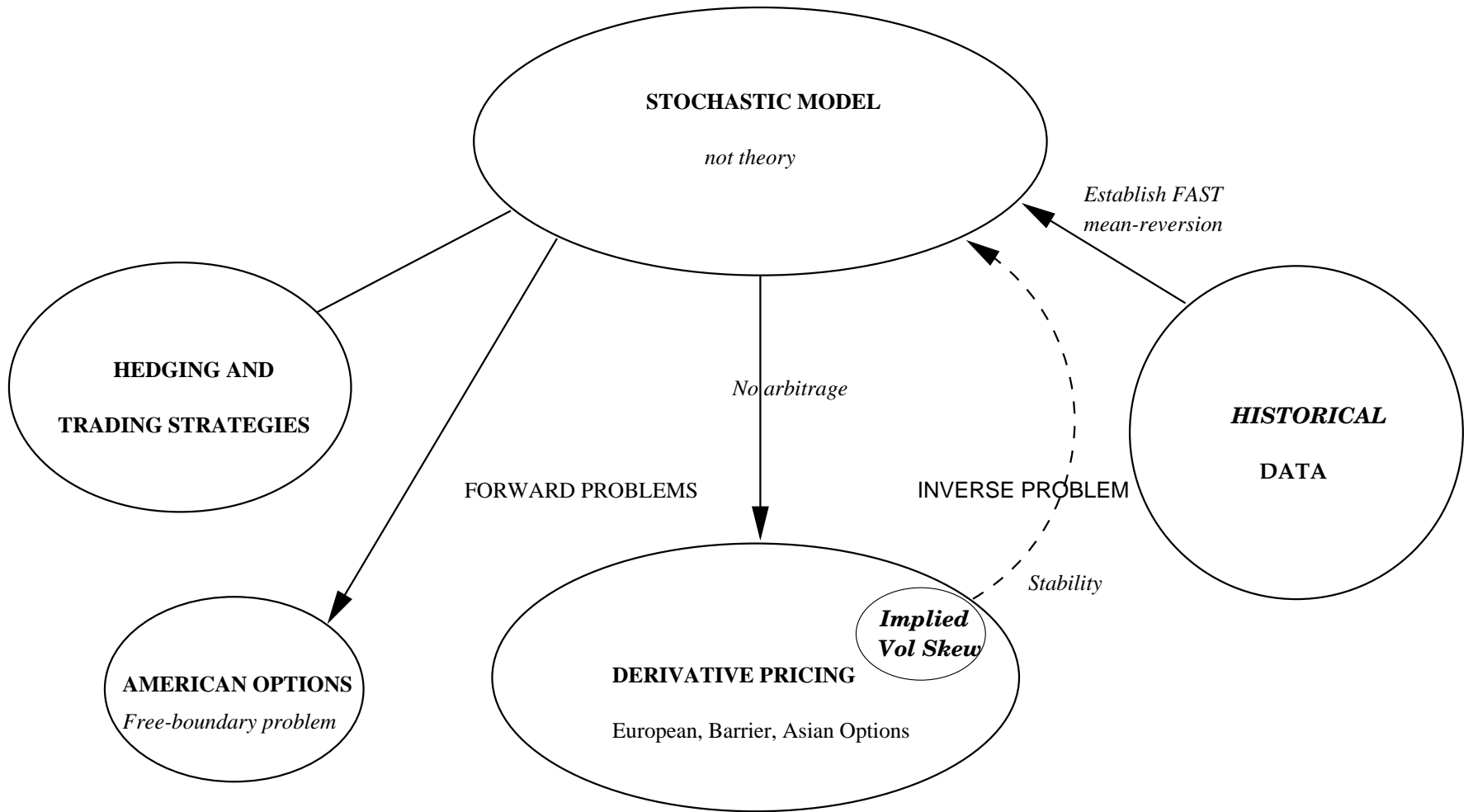
Implied Volatility Skew & Term-Structure



Basic Goals

- Parsimonious family of models that can reproduce **qualitatively** observed implied volatility surfaces.
- **Calibration** formulas that can shed some light on where data departs from the theory (*i.e.* things start going wrong with the models).
- Procedure to use calibrated parameters to value **exotic** derivatives contracts.
- **Stability** of estimated parameters over time.

(K, T, t) problem . Calibrate $K \text{ --- } T \text{ --- } t$.



Stochastic Volatility Models

- Volatility σ_t is a stochastic process

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t^{(0)},$$

introduced by Hull-White, Wiggins, Scott 1987.

- We'll take

$$\sigma_t = f(Y_t, Z_t),$$

where

Y **fast** volatility factor

Z **slow** volatility factor

Two-Scale Stochastic Volatility Models

$$\varepsilon, \delta \ll 1$$

$$dX_t = rX_t dt + f(Y_t, Z_t)X_t dW_t^{(0)\star}$$

$$dY_t = \left(\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t, Z_t) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)\star}$$

$$dZ_t = \left(\delta c(Z_t) - \sqrt{\delta} g(Z_t)\Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)\star}$$

$$d \langle W^{(0)\star}, W^{(1)\star} \rangle_t = \rho_1 dt$$

$$d \langle W^{(0)\star}, W^{(2)\star} \rangle_t = \rho_2 dt$$

Term Structure of Implied Volatility, Approx 1.0

$$I_0 + I_1^\varepsilon + I_1^\delta = \bar{\sigma} + b^\varepsilon + a^\varepsilon \frac{\log(K/x)}{T-t} + a^\delta \log(K/x) + b^\delta (T-t),$$

where the parameters $\bar{\sigma}$, $a^{\varepsilon,\delta}$ and $b^{\varepsilon,\delta}$ depend on Z_t and come from the primitives:

$$\bar{\sigma}(z)^2 = \langle f(\cdot, z)^2 \rangle$$

$$a^\varepsilon = \sqrt{\varepsilon} \rho_1 \frac{\nu}{\sqrt{2}\bar{\sigma}^3} \left\langle f \frac{\partial \phi}{\partial y} \right\rangle$$

$$b^\varepsilon = -a^\varepsilon \left(r - \frac{1}{2}\bar{\sigma}^2 \right) - \frac{\nu\sqrt{\varepsilon}}{\bar{\sigma}\sqrt{2}} \left\langle \Lambda_1 \frac{\partial \phi}{\partial y} \right\rangle$$

$$a^\delta = -\sqrt{\delta} \rho_2 \frac{g}{2\bar{\sigma}^3} \langle f \rangle \bar{\sigma}'$$

$$b^\delta = -a^\delta \left(r - \frac{1}{2}\bar{\sigma}^2 \right) + \sqrt{\delta} \frac{g}{2\bar{\sigma}} \langle \Lambda_2 \rangle \bar{\sigma}'$$

Fitting Procedure

Implied volatility approximation is of the form:

$$I \approx (a^\varepsilon + a^\delta \tau)(LMMR) + (\bar{\sigma} + b^\varepsilon + b^\delta \tau),$$

$$LMMR = \frac{\log(K/x)}{\tau}.$$

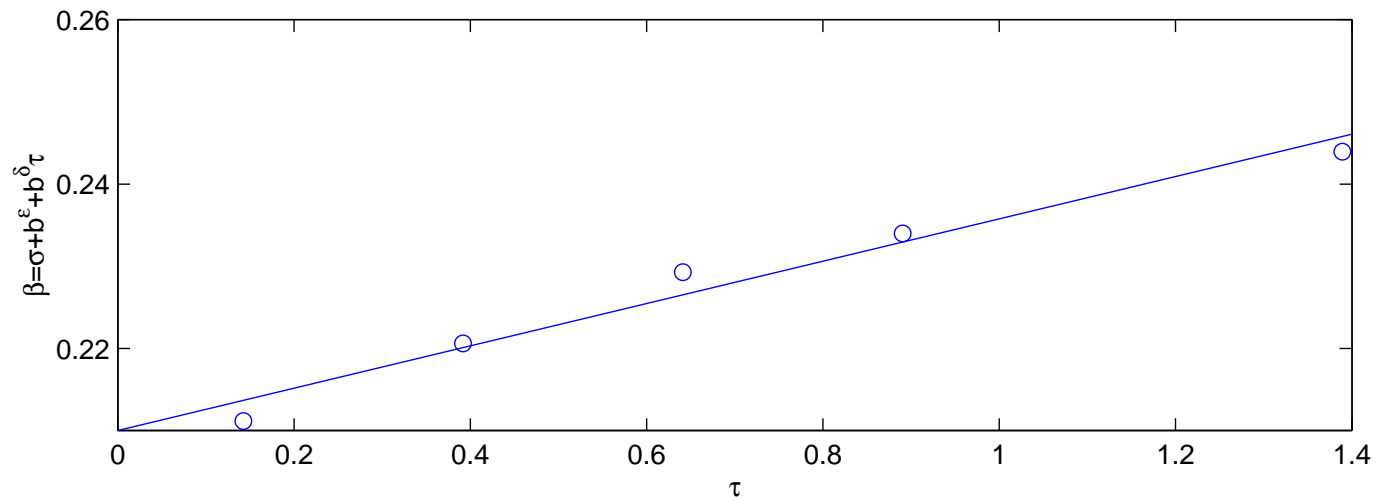
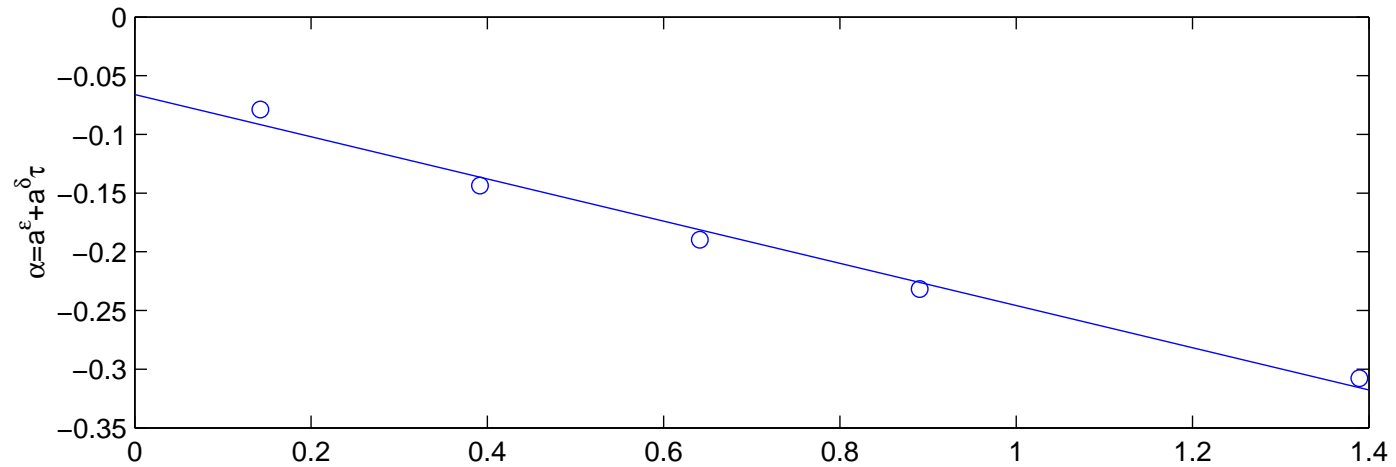
Suggests to fit skew to an affine function of LMMR for options of fixed maturity:

$$I = \alpha(\tau)LMMR + \beta(\tau)$$

and then the coefficients to affine functions of τ :

$$\alpha(\tau) = a^\varepsilon + a^\delta \tau \quad \beta(\tau) = \bar{\sigma} + b^\varepsilon + b^\delta \tau.$$

Term Structure Fit from S&P 500 Data, 5 June 2003



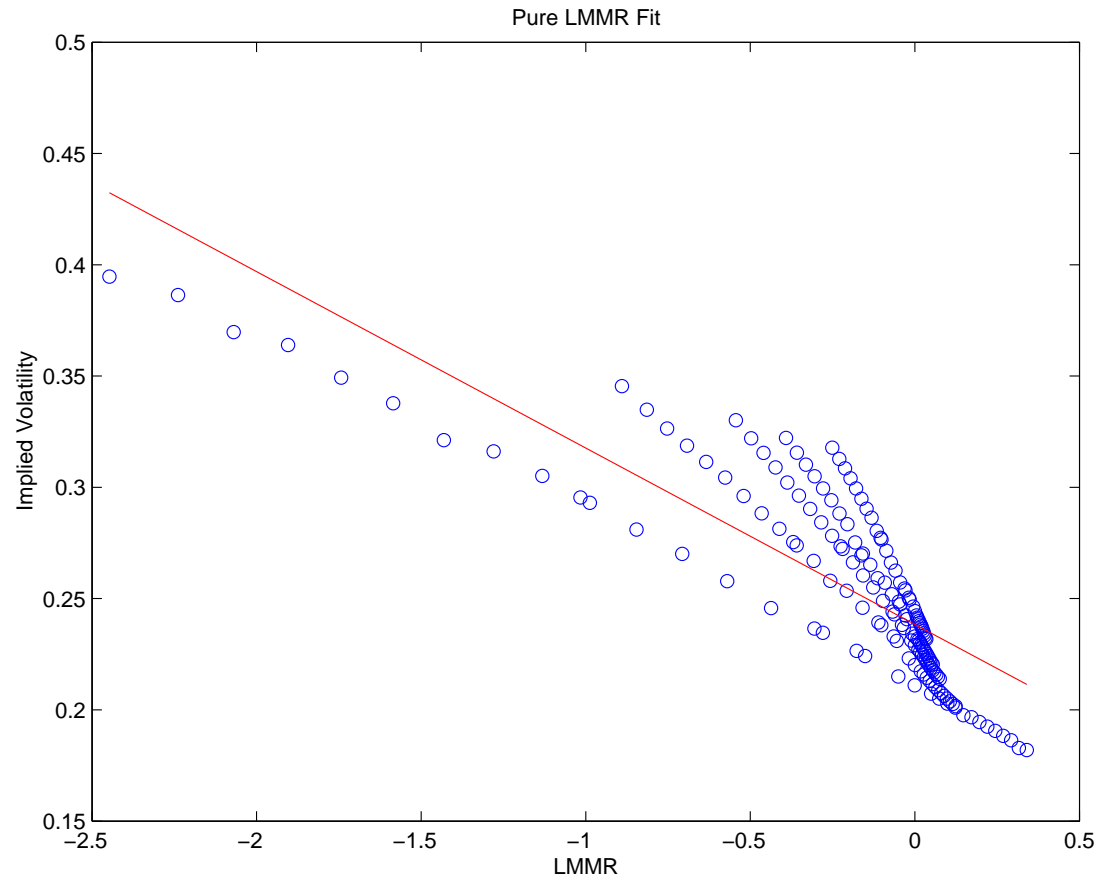


Figure 1: *S&P 500 implied volatilities as a function of LMMR on 25 January, 2000. The line $a^\varepsilon(\text{LMMR}) + b^\varepsilon + \bar{\sigma}$ shows the result using the estimated parameters from **only an LMMR (fast factor) fit**.*

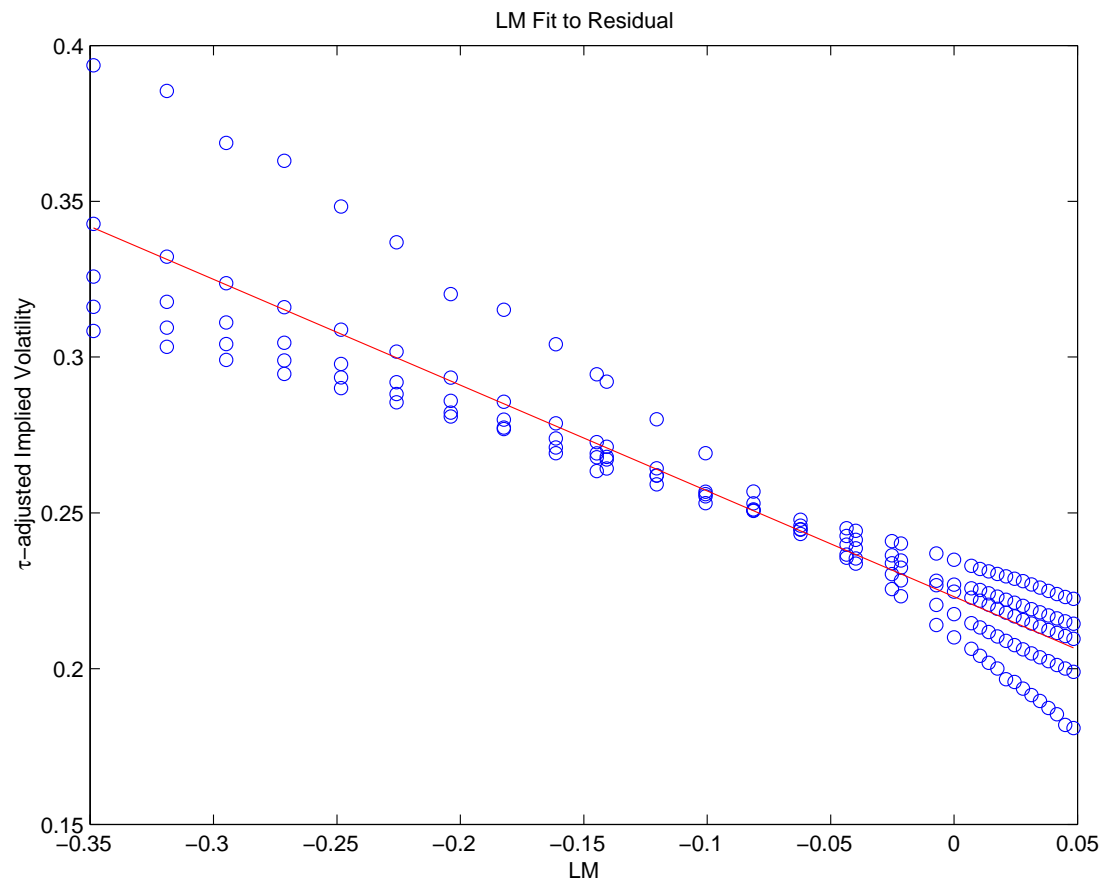


Figure 2: τ -adjusted implied volatility $I - b^\delta \tau$ as a function of LM . The circles are from *S&P 500* data, and the line $a^\delta(LM) + \bar{\sigma}$ shows the fit using the estimated parameters from *only a slow factor fit*.

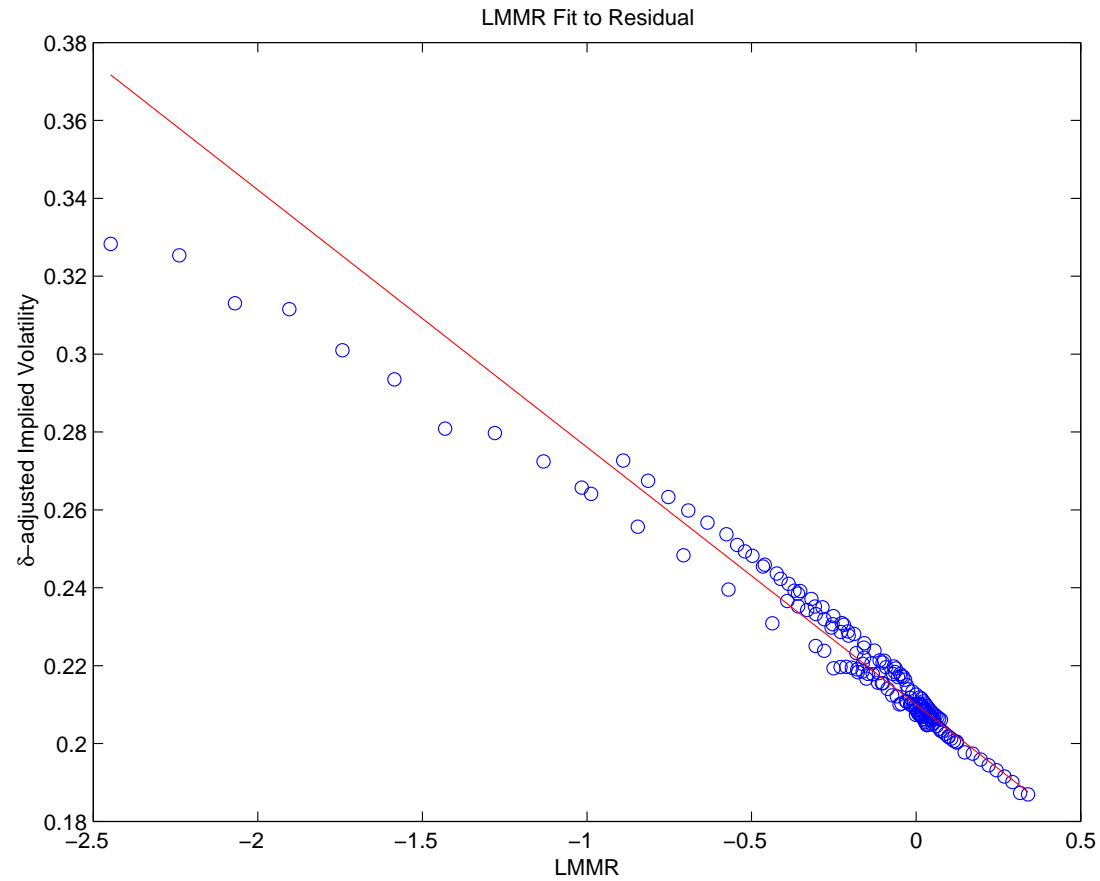


Figure 3: δ -adjusted implied volatility $I - b^\delta \tau - a^\delta(LM)$ as a function of LMMR. The circles are from S&P 500 data, and the line is from the full *fast* & *slow* factor fit.

Pricing other Derivative Securities

- Have calibrated $(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$, *linear transformations from* $(b^\delta, a^\delta, b^\varepsilon, a^\varepsilon)$, and $\bar{\sigma}$.
- The price of an exotic is approximated by $Q_{\text{BS}}(\bar{\sigma}) + \widetilde{Q}_1$, where the first term is the Black-Scholes price at volatility $\bar{\sigma}$.
- The correction satisfies $\mathcal{L}_{\text{BS}}(\bar{\sigma})\widetilde{Q}_1 =$

$$\left\{ \frac{2}{\bar{\sigma}} \left[V_0^\delta \frac{\partial}{\partial \sigma} + V_1^\delta x \frac{\partial^2}{\partial x \partial \sigma} \right] + V_2^\varepsilon x^2 \frac{\partial^2}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2}{\partial x^2} \right) \right\} Q_{\text{BS}}$$

with zero terminal condition and additional

- spatial (in x) boundary conditions: **barrier, lookback, passport options;**
- free boundary conditions: **American options;**
- state variables: **Asian options.**

Second Order Terms

Going further in the expansion:

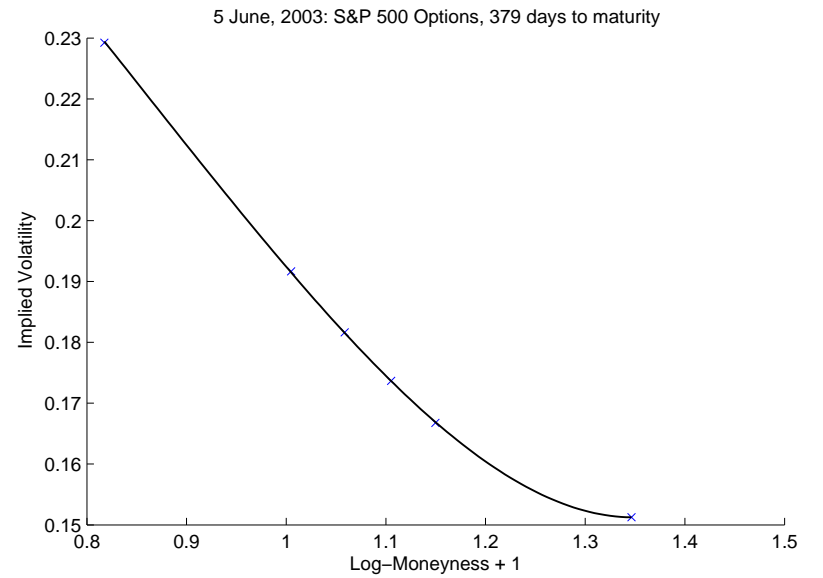
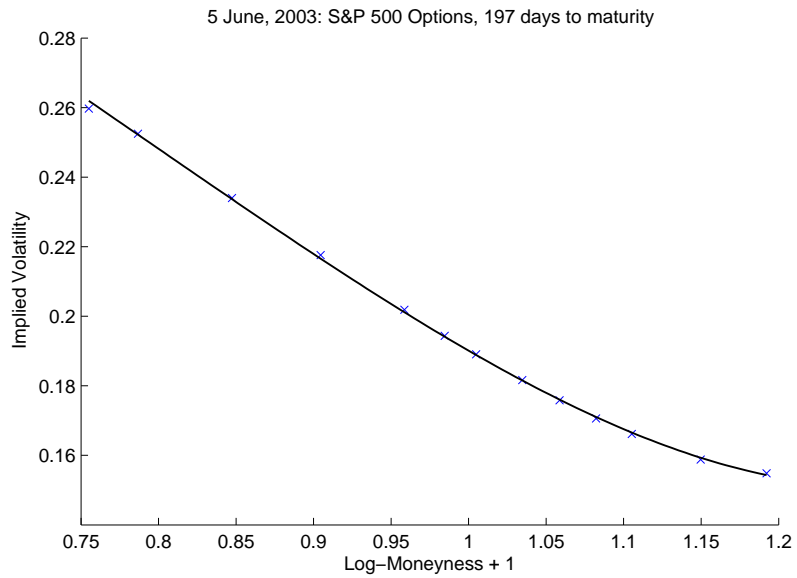
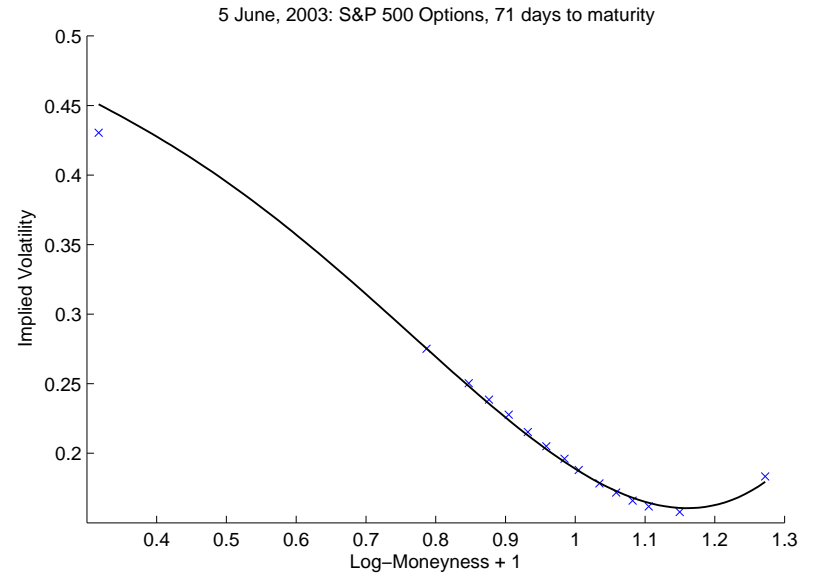
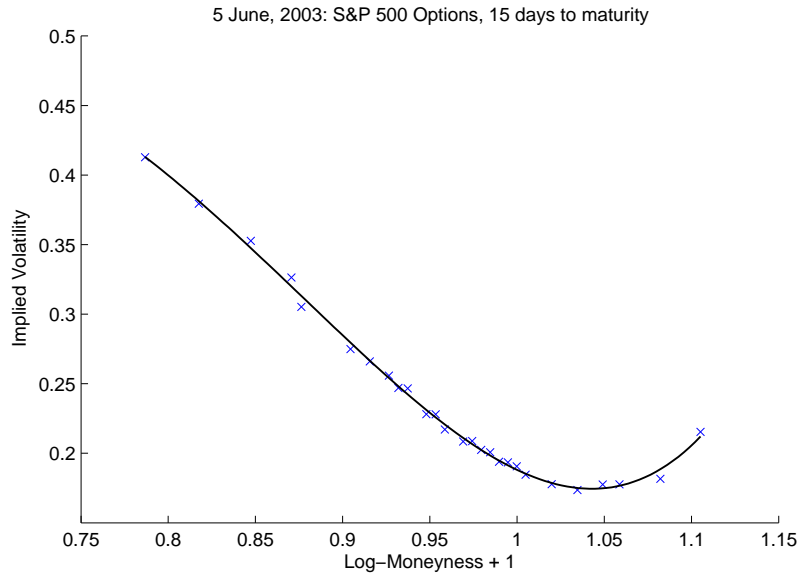
$$P^{\varepsilon, \delta} \approx P_{BS} + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \varepsilon P_{2,0} + \sqrt{\varepsilon \delta} P_{1,1} + \delta P_{0,2},$$

leads to an implied volatility approximation of the form

$$I \approx \sum_{j=0}^4 a_j(\tau) (\log(K/x))^j + \frac{1}{\bar{\sigma}\tau} c(Y_t).$$

Again, fit implied volatilities of fixed maturities to **quartic** in log-moneyness, then fit the coefficients $a_j(\tau)$ to (very complicated) functions of τ .

(Quadratic if correlation=0, FX).



Deeper into the Second Order δ Expansion

- The next significant term in the expansion is $\mathcal{O}(\delta)$.
- The group parameters that appear in the expansion up to $\mathcal{O}(\delta)$ are

$$\bar{\sigma}, V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon, F_0 := \frac{V_0'}{\bar{\sigma}'}, F_1 := \frac{V_1'}{\bar{\sigma}'},$$

$$F_2 := \sqrt{\delta} g \bar{\sigma}' \quad F_3 := \delta \left(\frac{1}{2} g^2 \bar{\sigma}'' + c \bar{\sigma}' \right).$$

The total is **nine** .

- From the first-order **linear** we already have $(\bar{\sigma}, V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$.
- The implied volatility contribution $I_{0,2}$ is

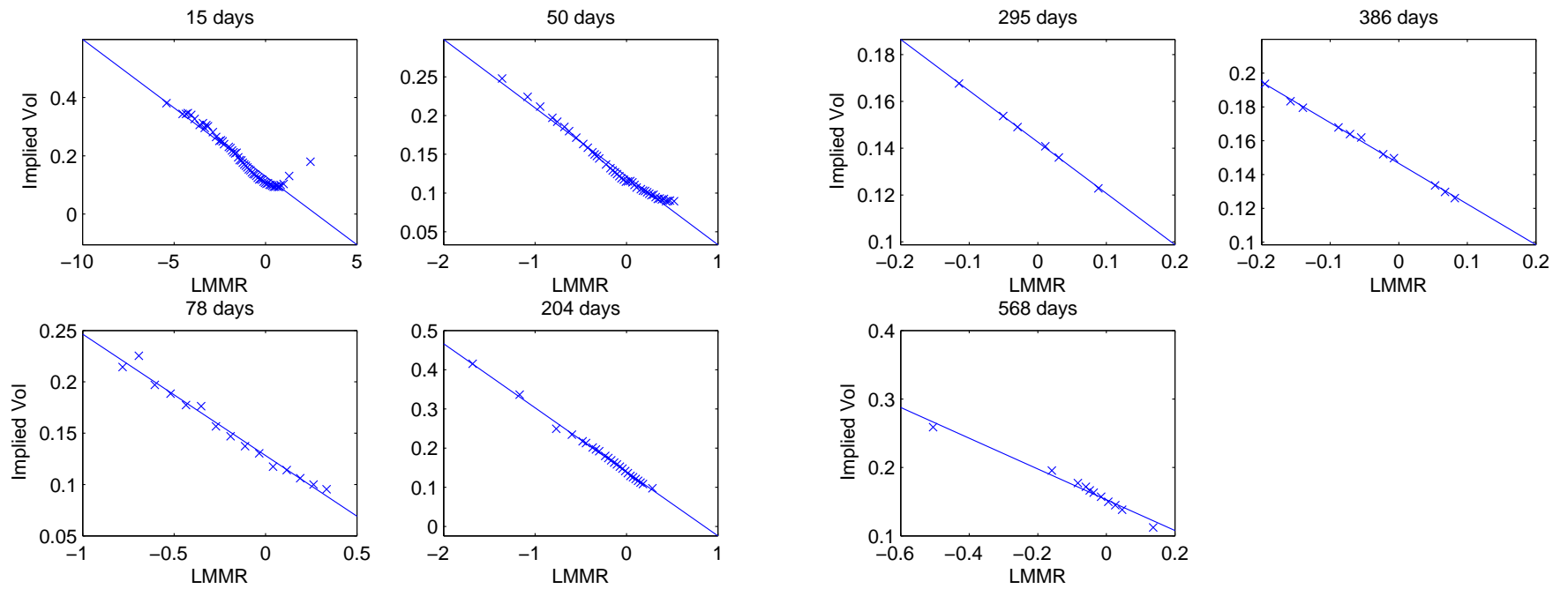
$$I_{0,2} = \sum_{j=1}^4 \tilde{c}_j(\tau) L^j,$$

where $L = \log(K/x)$.

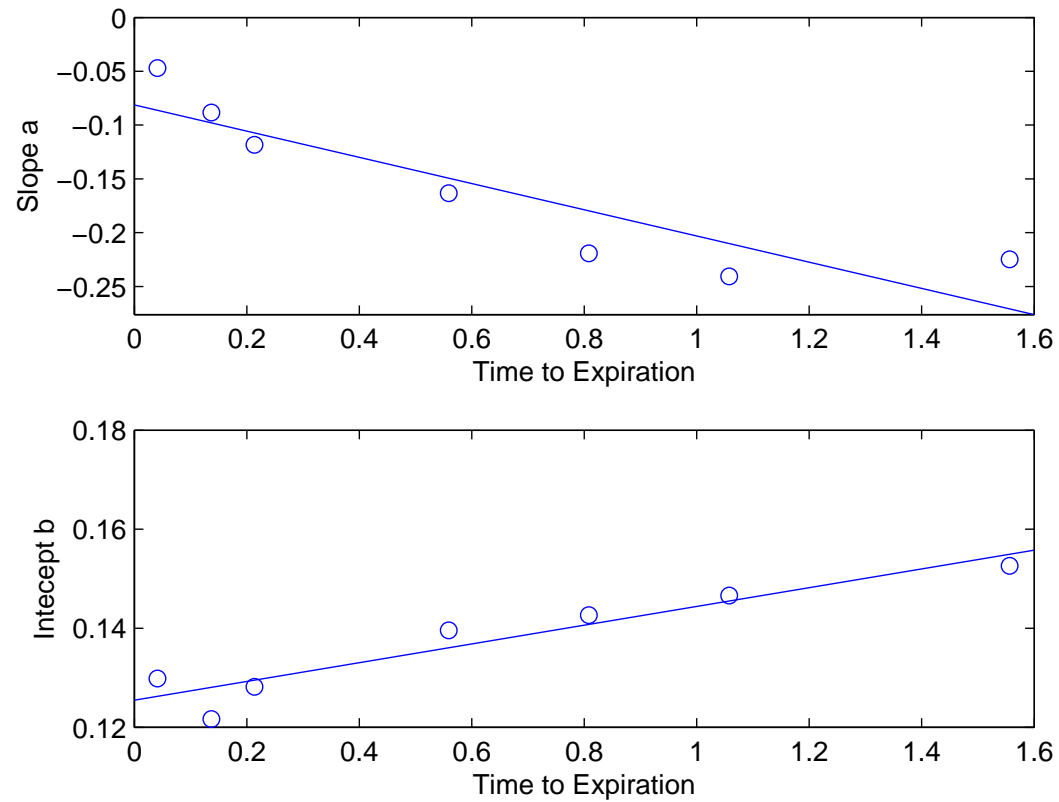
Calibration Constraints

- But $\tilde{c}_4(\tau) \sim \tau^{-1}$ and $\tilde{c}_3(\tau) = \text{const.}$ are **already determined** from the 1st-order fit.
- The residual is then to be fit to a **quadratic** in L .
- For the term-structure fit of \tilde{c}_2 , it should be **linear** in τ , but the slope is already known. The intercept gives a linear equation for F_1 and F_2^2 .
- For the term-structure fit of \tilde{c}_1 , it should be **quadratic** in τ with no constant, but the τ^2 coefficient is already known. The fitted τ coefficient gives a linear equation for F_1 , F_2^2 and F_0 .
- c_0 should be **cubic** in τ with no constant, but the τ^3 coefficient is already known. The fitted τ and τ^2 coefficient give two linear equations for F_1 , F_2^2 , F_3 and F_0 .
- Therefore, we have 4 equations for 4 unknowns: (F_0, F_1, F_2, F_3) .

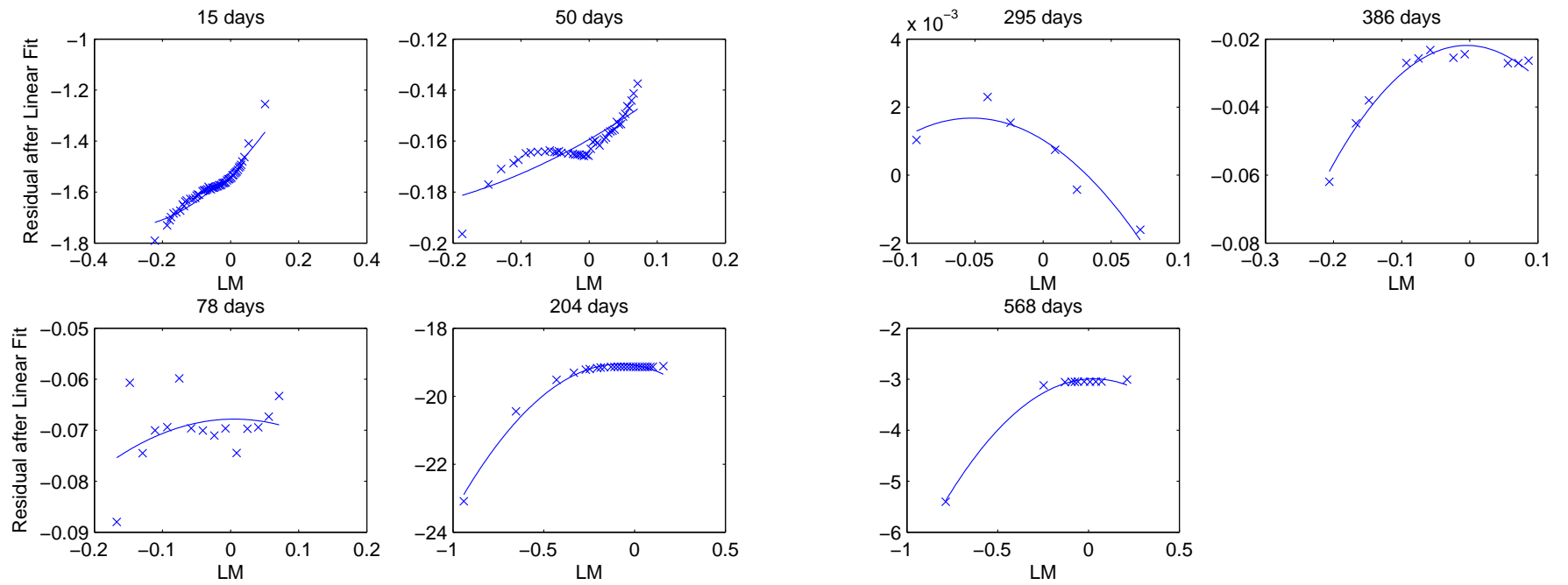
Linear LMMR Fit (1st-Order Theory): 1 June, 2007



Linear Term-Structure Fit (1st-Order Theory): 1 June, 2007



Quadratic Residual Fit (2nd-Order Theory): 1 June, 2007



Concluding Remarks

- The *first-order* theory works quite well across (K, T) . Stability in t OK too.
- The turn of the skew is relatively small and can be captured with the *second-order* theory, **but at a price** :
- Delicate calibration procedure needs refinement, and at second order it may be better to fit across K and T *together*.
- Need to compute Vega and Vega-Vega of exotics to compute their wing-adjusted second-order correction.
- Need a *maturity cycles* periodic term to take care of the shortest maturity.

Dealing with the Shortest Maturity

- Implied volatility skews steepen considerably as the time-to-maturity shortens.
- On a given day, the shortest maturity you see is as long as the time to the next expiration date. For S&P 500 options, this is every **third Friday of the month** .
- If we calibrate every day, there is a **missing data problem** : no information where the action is.
- However the expiration dates are pre-determined. Volatility often goes crazy as we approach these **witching dates** .
- Investigate this with the fast factor only theory:
$$I \approx a^\varepsilon \times \text{LMMR} + b^\varepsilon.$$

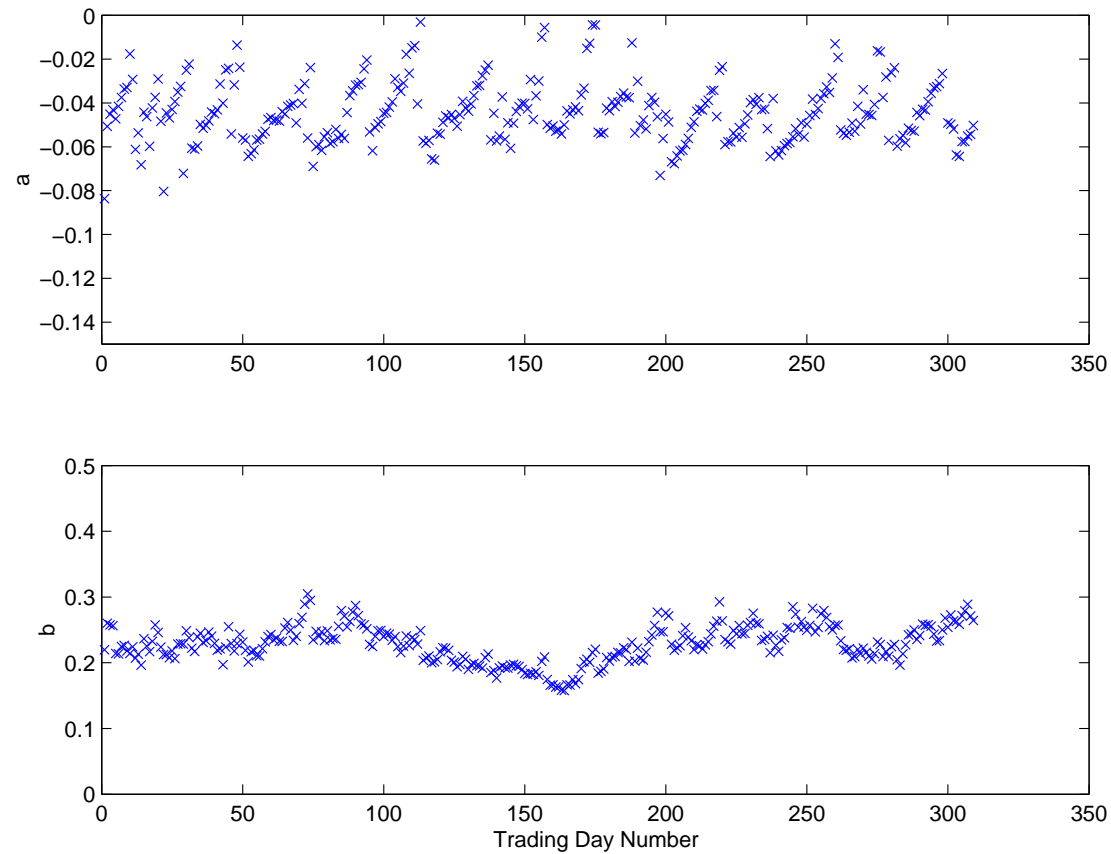


Figure 4: *Daily fitted LMMR parameters for options with at least 2 and at most 93 days to maturity, showing the periodicity corresponding to the monthly maturity cycle.*

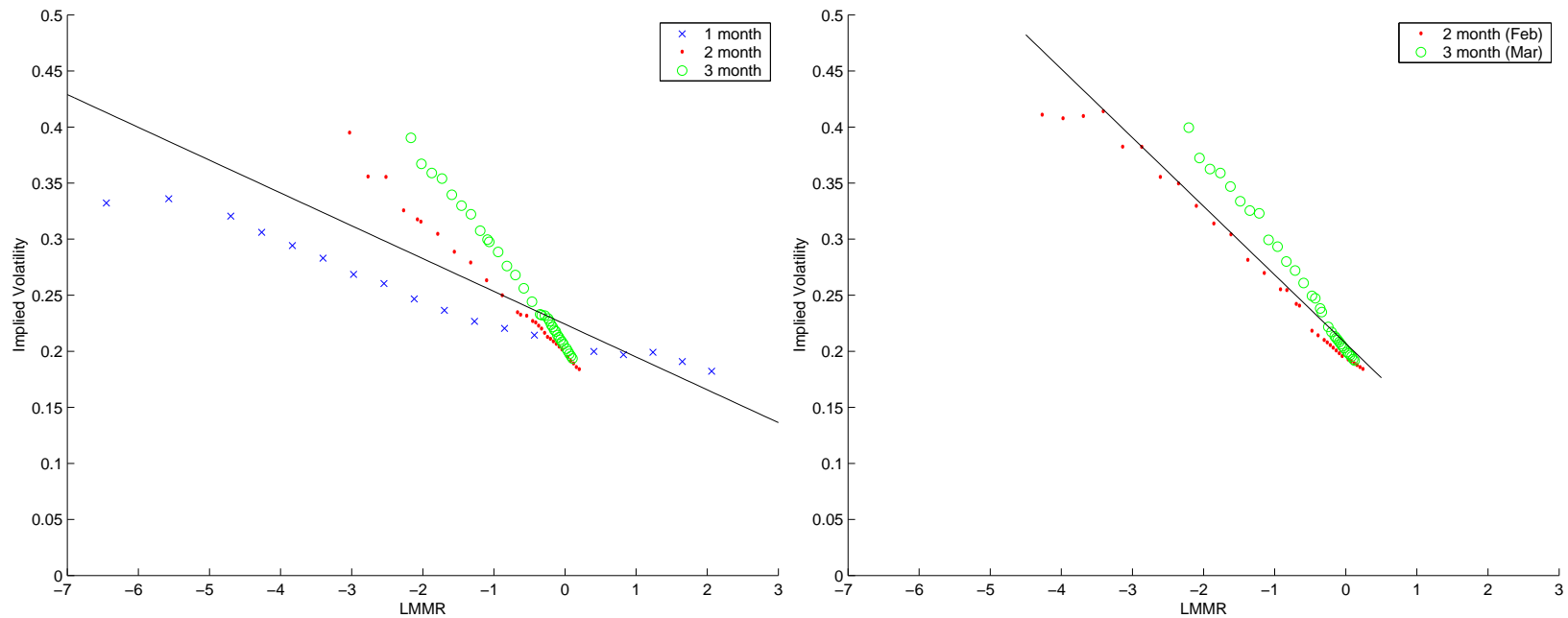


Figure 5: *In-sample LMMR fit the last day before a jump in the fitted slope (January 18, top) and after (January 19, bottom). Note that the shortest option has disappeared from data we are using in the fit.*

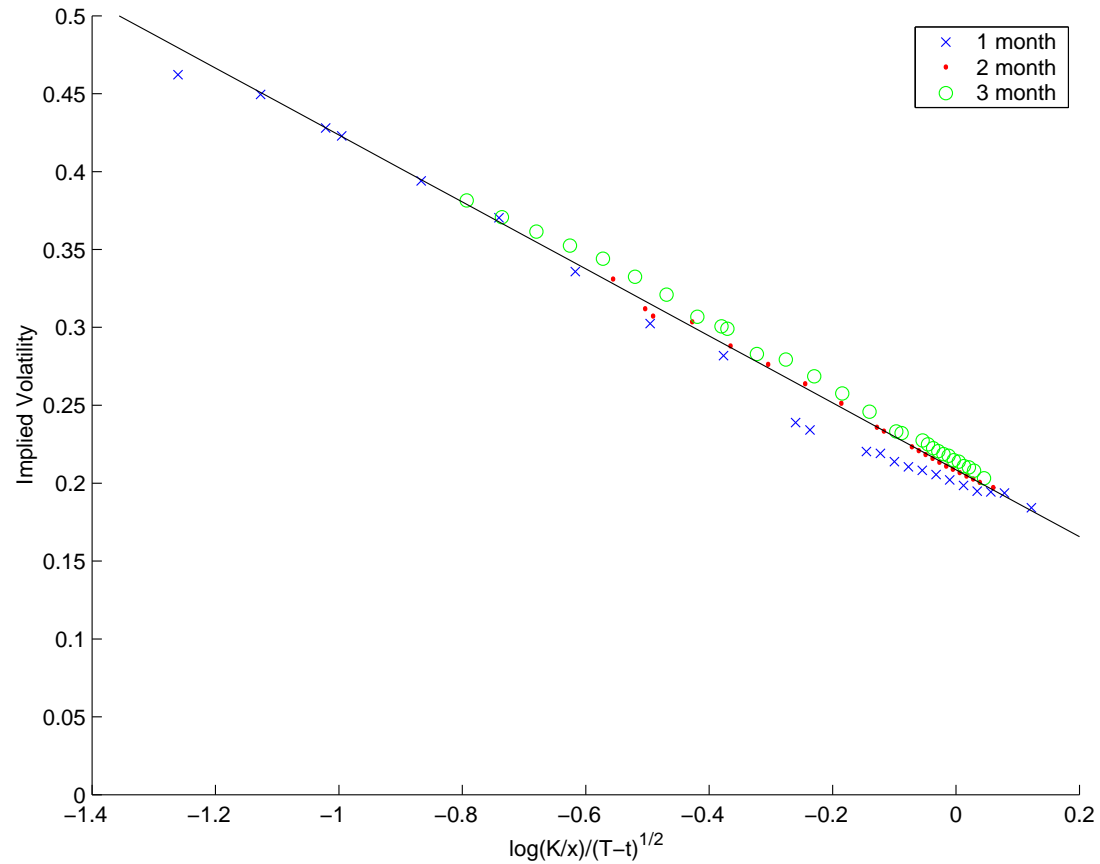


Figure 6: *Implied volatility as a function of $\log(K/X)/\sqrt{T-t}$ on January 12, 2000, for options with at least two and at most 93 days to maturity.*

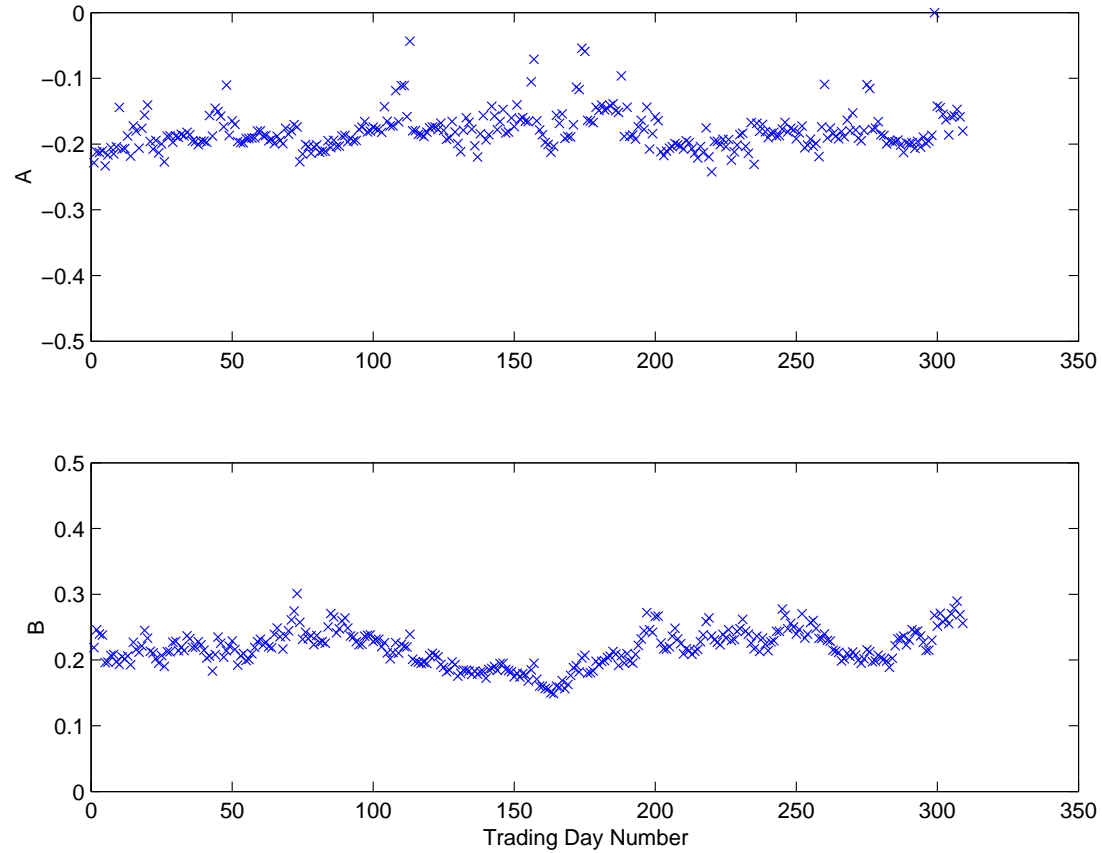


Figure 7: *Daily fitted slope A & intercepts B for options with at least 2 and at most 93 days to maturity fit to an affine function of $\log(K/X)/\sqrt{T-t}$.*

Time-dependent Theory

$$\begin{aligned}
 dX_t^\varepsilon &= rX_t^\varepsilon dt + f(Y_t^\varepsilon)X_t^\varepsilon dW_t^* \\
 dY_t^\varepsilon &= \left[\frac{1}{\varepsilon} \alpha(t)(m - Y_t^\varepsilon) - \frac{\nu \sqrt{2\alpha(t)}}{\sqrt{\varepsilon}} \Lambda(Y_t^\varepsilon) \right] dt + \frac{1}{\sqrt{\varepsilon}} \nu \sqrt{2\alpha(t)} d\hat{Z}_t^*,
 \end{aligned}$$

$$\frac{1}{\sqrt{\alpha(t)}} = c(T_{n(t)} - t)^{1/2},$$

for some constant c . Here $\{T_k\}$ are the [option maturity dates](#) (the third Friday of each month), and

$$n(t) = \inf\{n : T_n \geq t\},$$

the next expiration after time t .

Implied Volatility Approx. with Maturity Cycles

- Assume the expiration dates are evenly spaced:

$T_{k+1} - T_k =: \Delta T$. Decompose the time-to-maturity as $T - t = \Delta T(m_0 + \eta)$, with $0 \leq \eta < 1$. Then,

$$I \approx A \frac{\log(K/X)}{\mathcal{T}(t, T)} + B \frac{(T - t)}{\mathcal{T}(t, T)} + \bar{\sigma},$$

where the **effective time to maturity (ETM)** is

$$\mathcal{T}(t, T) = \frac{(T - t)}{\left(\frac{m_0 + \eta^{3/2}}{m_0 + \eta} \right)}.$$

- If $T - t < \Delta T$ then $m_0 = 0$ and $\eta = (T - t)/(\Delta T)$ so that

$$\mathcal{T}(t, T) \sim \sqrt{T - t},$$

whereas if $T - t \gg \Delta T$ then $\eta \ll m_0$, and therefore

$$\mathcal{T}(t, T) \sim (T - t).$$

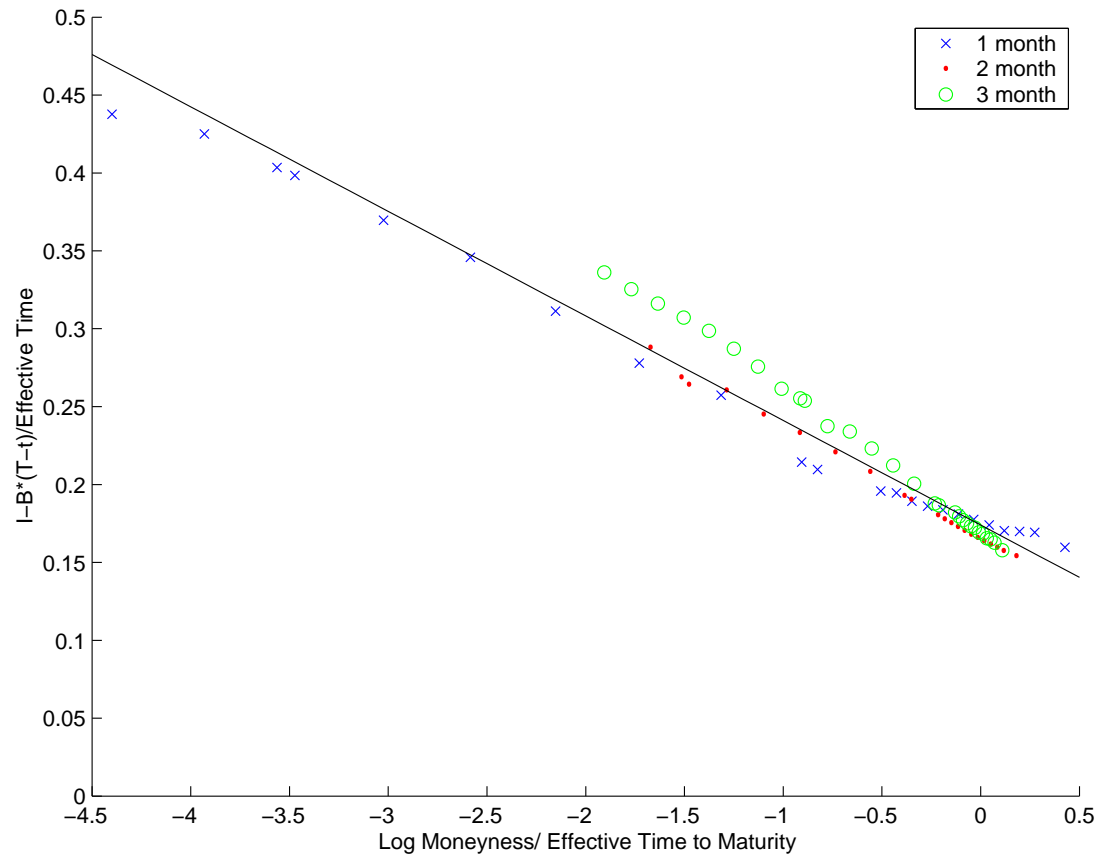


Figure 8: *Maturity-adjusted implied volatility as a function of log-moneyness to ETM.*

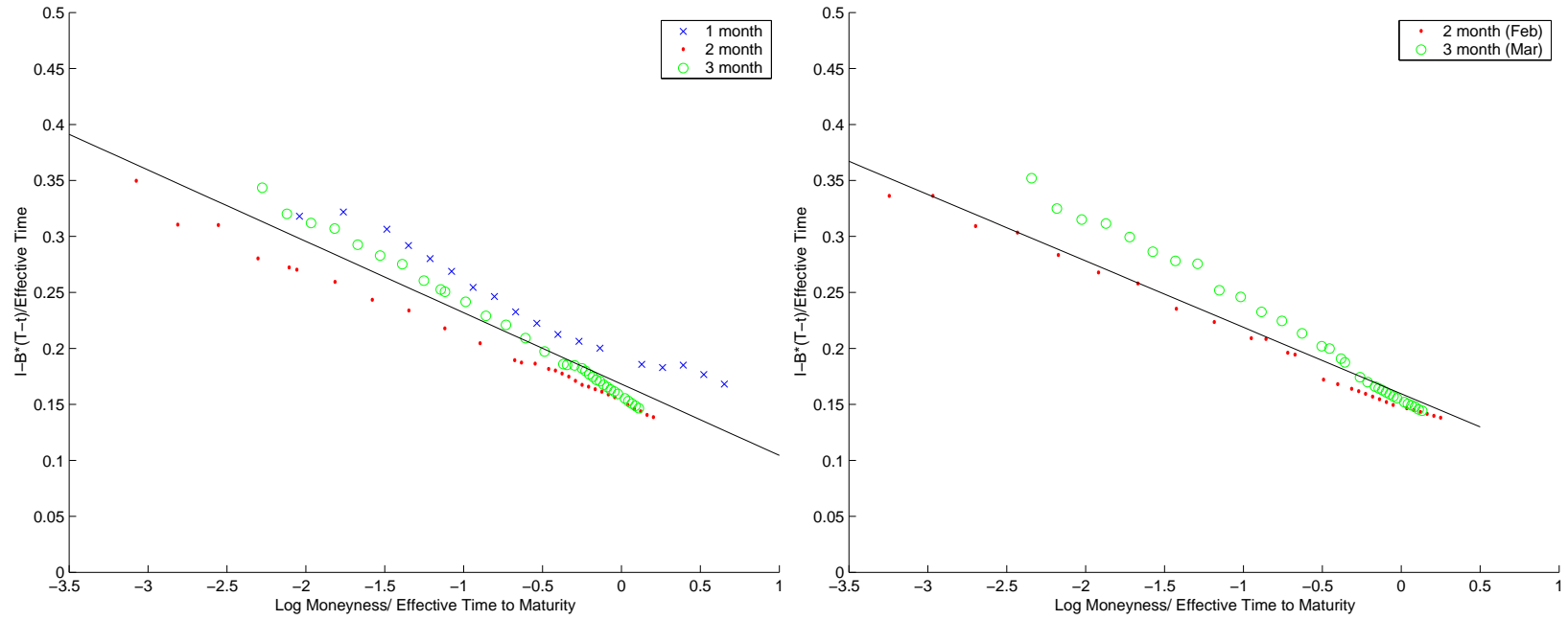


Figure 9: *Before and after.*

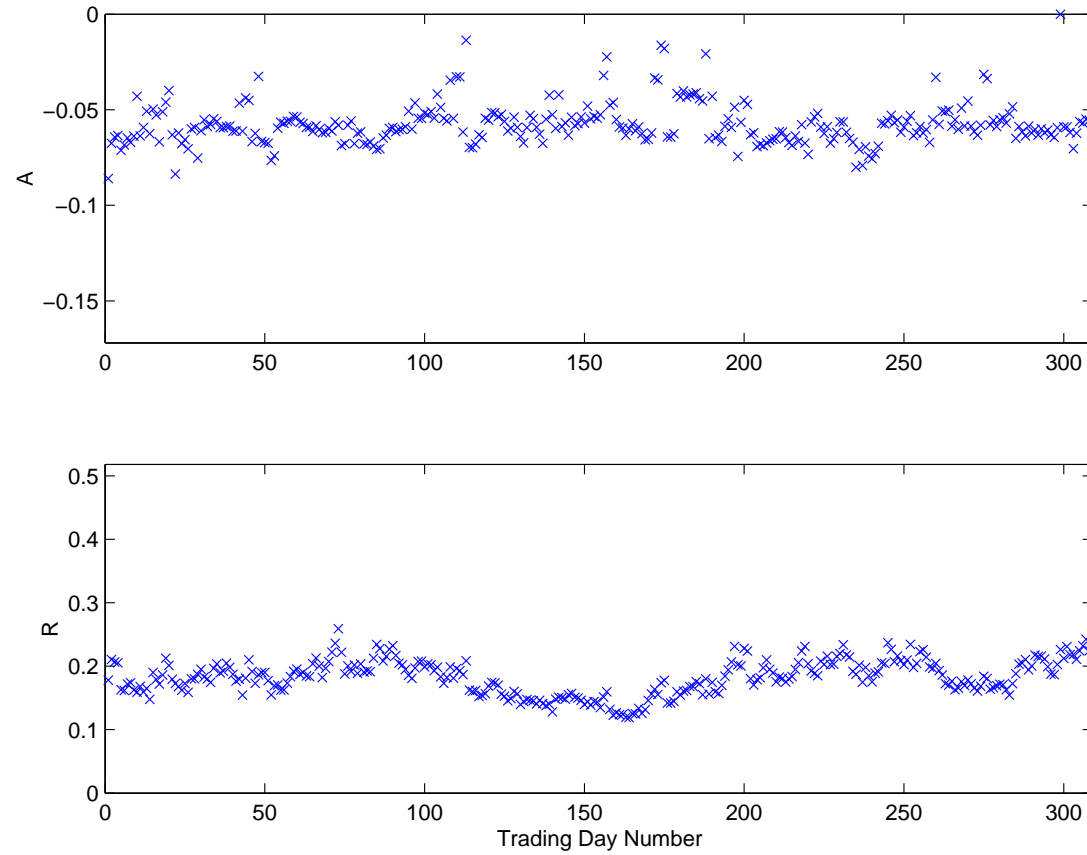


Figure 10: *Daily fitted slope A & residual R for options with at least 2 and at most 93 days to maturity fit to ETM formula.*