

A Futures Trading Model with Transaction Costs

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- ▶ Cash in money market (total capital):

$$dX(t) = Y(t) dF(t) - \lambda(dL(t) + dM(t)) + rX(t) dt - c(t)X(t) dt.$$

Parameter assumptions:

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Introduce trading proportional to total capital:

$$dL(t) = X(t-) dl(t), \quad dM(t) = X(t-) dm(t).$$

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$$v(x, y) \triangleq \sup_{\ell, m, c} \mathbb{E} \int_0^{\infty} e^{-\beta t} U(c(t)X(t)) dt \quad \forall (x, y) \in \mathcal{S}.$$

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- ▶ Homotheticity: For $\gamma > 0$,

$$v(\gamma x, \gamma y) = \gamma^{1-p} v(x, y).$$

If transaction cost were zero

If λ were zero, the value function would be

$$v_0(x) = \frac{1}{1-p} A^{-p} x^{1-p} \quad \forall x \geq 0,$$

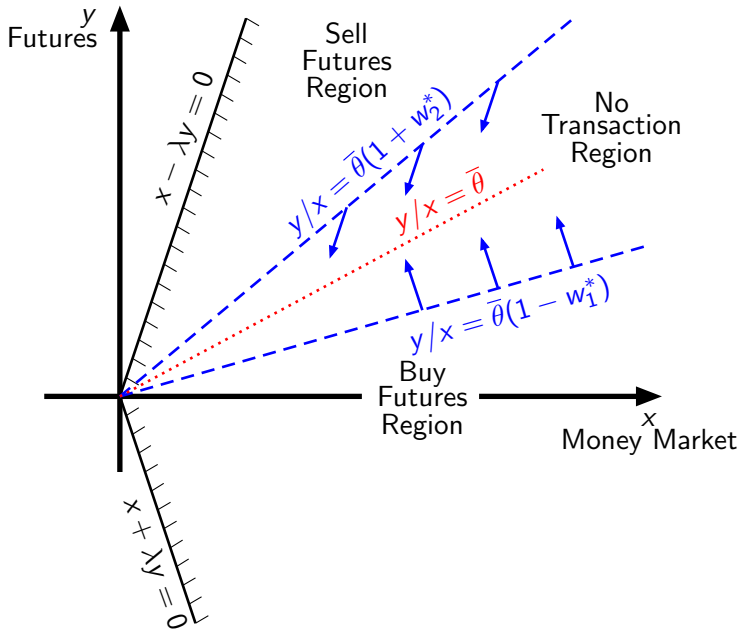
where

$$A \triangleq \frac{\beta - r(1-p)}{p} - \frac{\alpha^2(1-p)}{2\sigma^2 p^2}.$$

The optimal investment proportion would be

$$\frac{Y(t)}{X(t)} = \bar{\theta} \triangleq \frac{\alpha}{\sigma^2 p}.$$

The optimal consumption proportion would be $c(t) \equiv A$.



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Reasons to do this:

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Reasons to do this:
 - (a) Attempt to escape the Markov bonds.
 - (b) Probability is more fun.

Outline of approach

For a arbitrary control policy, define the investment proportion

$$\theta(t) \triangleq \frac{Y(t)}{X(t)}.$$

The optimal consumption is a function $c^*(\theta^*(t))$ of the optimal investment proportion $\theta^*(\cdot)$.

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Evolution of $\theta(t)$:

$$d\theta(t) = \theta(t)(-r + c(\theta(t)) - \alpha\theta(t) + \sigma^2\theta^2(t)) dt - \sigma\theta^2(t) dW(t) \\ + (1 + \lambda\theta(t)) d\ell(t) - (1 - \lambda\theta(t)) dm(t).$$

The optimal $\theta^(\cdot)$ is a doubly-reflected diffusion in $[\bar{\theta}(1 - w_1^*), \bar{\theta}(1 + w_2^*)]$.*

We know the form of the optimal policy. We only need to find w_1^* , w_2^* and $c^*(\cdot)$.

Outline of approach (continued)

To find w_1^* , w_2^* and $c^*(\cdot)$:

- ▶ Fix a function $c(\cdot)$, and “small” positive numbers w_1 and w_2 .
- ▶ Let $\theta(t)$ be the corresponding diffusion with reflection at the boundaries of $[\bar{\theta}(1 - w_1), \bar{\theta}(1 + w_2)]$.
- ▶ Estimate the expected utility associated with this consumption and investment strategy.
- ▶ Optimize over w_1 , w_2 and $c(\cdot)$. This involves balancing **displacement loss** and **transaction loss**.

The investment ratio process

Fix $c(\cdot)$, w_1 and w_2 . Define $\theta(t)$ by

$$d\theta(t) = \theta(t)(-r + c(\theta(t)) - \alpha\theta(t) + \sigma^2\theta^2(t)) dt - \sigma\theta^2(t) dW(t) \\ + d\bar{\ell}(t) - d\bar{m}(t).$$

where $\bar{\ell}$ and \bar{m} cause reflection at the boundaries of $[\bar{\theta}(1 - w_1), \bar{\theta}(1 + w_2)]$.

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The pushing processes $\bar{\ell}$ and \bar{m} that cause θ to reflect will be related to trading processes ℓ and m by formulas

$$(1 + \lambda\bar{\theta}(1 - w_1))\ell(t) = \bar{\ell}(t), \quad (1 - \lambda\bar{\theta}(1 + w_2))m(t) = \bar{m}(t),$$

but for now we have only the process $\theta(t)$, not the processes $X(t)$, $Y(t)$ nor the trading processes ℓ and m .

Displacement loss

One can show a priori that

$$c^*(\cdot) = \text{Constant} + O(\lambda).$$

This permits us to replace $c(\cdot)$ by a **constant** c and then optimize over the **constant** c .

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Choose a constant c . Take $X_0(0) = X_1(0) = 1$.

- ▶ Consider **investment ratio** $\bar{\theta}$ and no transaction cost:

$$dX_0(t) = X_0(t) [(r - c + \alpha \bar{\theta}) dt + \sigma \bar{\theta} dW(t)],$$

$$u_0 = \mathbb{E} \int_0^\infty e^{-\beta t} U(cX_0(t)) dt.$$

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- ▶ Consider **investment ratio $\theta(\cdot)$** and no transaction cost:

$$\begin{aligned}dX_1(t) &= X_1(t) [(r - c + \alpha\theta(t)) dt + \sigma\theta(t) dW(t)], \\u_1 &= \mathbb{E} \int_0^\infty e^{-\beta t} U(cX_1(t)) dt.\end{aligned}$$

Theorem

The *displacement loss* is

$$u_0 - u_1 = \frac{u_0 p (1 - p) \sigma^2 \bar{\theta}^2}{6(pA + (1 - p)c)} (w_1^2 - w_1 w_2 + w_2^2) + O((w_1 + w_2)^3).$$

METHOD OF PROOF: Use the maximal martingale inequality and a change of measure trick due to Chris Rogers.

Remark

In the case $w_1 = w_2 \triangleq w$, The displacement loss is $O(w^2)$, which is increasing in w .

Outline of displacement loss proof

Compute directly

$$\begin{aligned} X_1^{1-p}(t) &= \bar{Z}(t) \cdot \mathbb{E}X_0^{1-p}(t) \\ &\quad \cdot \exp \left\{ (1-p)\sigma \int_0^t (\theta(u) - \bar{\theta}) d\bar{W}(u) \right. \\ &\quad \left. - \frac{1}{2}(1-p)\sigma^2 \int_0^t (\theta(u) - \bar{\theta})^2 du \right\}, \end{aligned}$$

where

$$\begin{aligned} \bar{Z}(t) &= \exp \left\{ (1-p)\sigma\bar{\theta}W(t) - \frac{1}{2}(1-p)^2\sigma^2\bar{\theta}^2 t \right\}, \\ \bar{W}(t) &= W(t) - (1-p)\sigma\bar{\theta}t. \end{aligned}$$

Then

$$\mathbb{E}X_1^{1-p}(t) = \mathbb{E}X_0^{1-p}(t) \bar{\mathbb{E}} \exp \left\{ \dots \dots \right\}.$$

Outline of displacement loss proof (continued)

Furthermore,

$$d\theta(t) = (\text{Bounded drift}) dt - \sigma\theta^2(t) d\bar{W}(t) + d\bar{\ell}(t) - d\bar{m}(t).$$

Asymptotically as $w_1 \downarrow 0$ and $w_2 \downarrow 0$, $\theta(t)$ is uniformly distributed in $[\bar{\theta}(1 - w_1), \bar{\theta}(1 + w_2)]$ under $\bar{\mathbb{P}}$. Therefore, we can compute the needed terms in the asymptotic expansion in w_1 and w_2 of

$$\bar{\mathbb{E}} \exp \left\{ (1 - \rho)\sigma \int_0^t (\theta(u) - \bar{\theta}) d\bar{W}^T(u) - \frac{1}{2}(1 - \rho)\sigma^2 \int_0^t (\theta(u) - \bar{\theta})^2 du \right\}.$$

Transaction cost loss

- ▶ We just considered investment ratio $\theta(\cdot)$ and **no transaction cost**:

$$dX_1(t) = X_1(t) [(r - c + \alpha\theta(t)) dt + \sigma\theta(t) dW(t)],$$
$$u_1 = \mathbb{E} \int_0^\infty e^{-\beta t} U(cX_1(t)) dt.$$

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$$u_1 = \mathbb{E} \int_0^\infty e^{-\beta t} U(cX_1(t)) dt.$$

- ▶ Now consider investment ratio $\theta(\cdot)$ and **payment of transaction cost**:

$$dX_2(t) = X_2(t) \left[(r - c + \alpha\theta(t)) dt + \sigma\theta(t) dW(t) \right. \\ \left. - \lambda(d\ell(t) + dm(t)) \right],$$
$$u_2 = \mathbb{E} \int_0^\infty e^{-\beta t} U(cX_2(t)) dt.$$

Theorem

The *transaction loss* is

$$u_1 - u_2 = \frac{u_0(1-p)\sigma^2\bar{\theta}^3}{pA + (1-p)c} \cdot \frac{\lambda}{w_1 + w_2} + O(\lambda) + O\left(\frac{\lambda^2}{(w_1 + w_2)^2}\right).$$

METHOD OF PROOF: Detailed analysis of the “pushing processes” $\ell(t)$ and $m(t)$.

Remark

The *transaction cost* is decreasing in w_1 and w_2 .

Outline of transaction loss proof

We have

$$\begin{aligned} X_1^{1-p}(t) - X_2^{1-p}(t) &= X_1^{1-p}(t) \left[1 - e^{-\lambda(1-p)(\ell(t)+m(t))} \right] \\ &= \tilde{Z}(t) \cdot \exp \left\{ (1-p)\sigma \int_0^t \left(r - c + \alpha\theta(u) - \frac{1}{2}p\sigma^2\theta^2(u) \right) du \right\} \\ &\quad \cdot \left[1 - e^{-\lambda(1-p)(\ell(t)+m(t))} \right] \\ &\approx \tilde{Z}(t) \cdot \exp \left\{ (1-p)\sigma \int_0^t \left(r - c + \alpha\bar{\theta} - \frac{1}{2}p\sigma^2\bar{\theta}^2 \right) du \right\} \\ &\quad \cdot \left[1 - e^{-\lambda(1-p)(\ell(t)+m(t))} \right], \end{aligned}$$

where

$$\tilde{Z}(t) = \exp \left\{ (1-p)\sigma \int_0^t \theta(u) dW(u) - \frac{1}{2}(1-p)^2\sigma^2 \int_0^t \theta^2(u) du \right\}.$$

Outline of transaction loss proof (continued)

Estimate

$$\begin{aligned} & \mathbb{E} \left[\tilde{Z}(t) \left(1 - e^{-\lambda(1-\rho)(\ell(t)+m(t))} \right) \right] \\ &= \tilde{\mathbb{E}} \left[\left(1 - e^{-\lambda(1-\rho)(\ell(t)+m(t))} \right) \right] \\ &= \lambda(1-\rho)\tilde{\mathbb{E}}[\ell(t) + m(t)] + \text{Remainder}. \end{aligned}$$

Show that

$$\lambda(1-\rho)\tilde{\mathbb{E}}[\ell(t) + m(t)] \approx \frac{(1-\rho)\sigma^2\bar{\theta}^3 \lambda t}{w_1 + w_2},$$

and “Remainder” is negligible.

Main result

Theorem

For initial capital 1, the optimal value in the problem with positive transaction cost λ is

$$v_0(1) - \frac{\sigma^2 \bar{\theta}^{8/3}}{A^{1+p}} \left(\frac{9p}{32} \right)^{1/3} \lambda^{2/3} + O(\lambda^{5/6}),$$

where v_0 is the value in the problem with zero transaction cost.

METHOD OF PROOF: Minimize sum of losses.

Outline of proof of main result

Sum of largest terms in displacement loss and transaction loss is

$$C_1(w_1^2 - w_1 w_2 + w_2^2) + \frac{C_2 \lambda}{w_1 + w_2}. \quad (\text{Total Loss})$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial w_1} \text{Total Loss} &\approx C_1(2w_1 - w_2) - \frac{C_2 \lambda}{(w_1 + w_2)^2}, \\ \frac{\partial}{\partial w_2} \text{Total Loss} &\approx C_1(2w_2 - w_1) - \frac{C_2 \lambda}{(w_1 + w_2)^2}. \end{aligned}$$

Set derivatives equal to zero to get

$$w_1 \approx w_2 \approx \left(\frac{C_2 \lambda}{4C_1} \right)^{1/3} = O(\lambda^{1/3}).$$

Substitute into (Total Loss) to see that total loss is $O(\lambda^{2/3})$.