Moment Explosions and Stationary Distributions in Affine Diffusion Models

Kyoung-Kuk Kim

(Joint work with Paul Glasserman)

Columbia Business School

November 30, 2007

Table of Contents

Introduction

Affine Diffusion Models

$$dY(t) = \mu(t)dt + \sigma(t)dW(t)$$

$\mu(t)$ and $\sigma(t)\sigma(t)^\top$ are affine functions of $Y(t)$

Why affine models? Modeling flexibility + Tractability

Examples

- Vasicek: $dY(t) = \alpha(b - Y(t))dt + \sigma dW(t)$
- Cox-Ingersoll-Ross: $dY(t) = \alpha(b - Y(t))dt + \sigma\sqrt{Y(t)}dW(t)$
- Heston: $W(t)$ 2-dimensional Brownian motion with correlation ρ, $S(t) = \exp(Y_2(t))$

$$\begin{aligned} dY_1(t) &= \alpha(b - Y_1(t))dt + \sigma\sqrt{Y_1(t)}dW_1(t) \\ dY_2(t) &= (r - Y_1(t)/2)dt + \sqrt{Y_1(t)}dW_2(t) \end{aligned}$$

Transform Formula

1. Tractability of Affine Models

   –Semi-analytical Bond or Option pricing formula via Fourier inversion

2. Duffie, Pan, Singleton, Econometrica 2000

   –Affine Jump-Diffusion Models under regularity conditions

$$\boxed{\mathbb{E}\exp(u \cdot Y(t)) = \exp\left(x(t) + y(t) \cdot Y(0)\right)}$$

   –$x(t)$, $y(t)$: solutions of a system of ODEs

3. Duffie, Filipović, Schachermayer, Annals of App. Prob. 2003

   –Admissible parameters $\Rightarrow$ Fourier transform holds

Objectives

1. Moment Explosions (Andersen and Piterbarg, Fin. and Stoch. 07)

   –Moment explosions for SV models

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \epsilon V_t^p dW_t^1 \\ dX_t &= \lambda X_t \sqrt{V_t} dW_t^2 \end{aligned}$$

2. Tail Behavior of $u \cdot Y_t$: $\phi(\theta) := \mathbb{E}e^{\theta X}$, $X \geq 0$ a.s. and $\theta \geq 0$

   - If $\phi(\theta) = \infty$ for all $\theta > 0$, X is heavy-tailed
   - If $\phi(\theta) < \infty$ for $\theta < \theta_0$, X has exponentially bounded tails
   - If $\phi(\theta) < \infty$ for all θ, X is light-tailed

3. Limiting stationary distribution Y_∞

Canonical Affine Diffusions

Canonical Models

Dai and Singleton, Journal of Finance 2000
(econometric identifiability)

Canonical Affine Diffusions: restrictions on parameters to identify $r_t = a + b \cdot V_t + c \cdot X_t$

–Volatility factors (dimension m)

$$\boxed{dV_t = K_1(\Theta_1 - V_t)dt + \sqrt{diag(V_t)}dW_t^1}$$

–Dependent factors (dimension n)

$$\boxed{dX_t = (K_2(\Theta_2 - X_t) - K_3 V_t)\, dt + \sqrt{diag(1 + MV_t)}dW_t^2}$$

$$\Longrightarrow \mathbb{A}_m(m+n)$$

Theorem 1

For given $u \in \mathbb{R}^{m+n}$ and $t > 0$ ($Y = (V, X)$),

$$\mathbb{E}\exp(u \cdot Y_t) < \infty? \Leftrightarrow |x(t)| < \infty?$$

$$\dot{x} = Ax + B \begin{pmatrix} x_1^2 \\ \dots \\ x_{n+m}^2 \end{pmatrix}, \quad x(0) = u/2$$

$$A = -\begin{pmatrix} K_1^\top & K_3^\top \\ 0 & K_2^\top \end{pmatrix}, \quad B = \begin{pmatrix} I & M^\top \\ 0 & 0 \end{pmatrix} \geq 0$$

$$\boxed{\mathbb{E}\exp(u \cdot Y_t) = \exp\left(2\int_0^t \Lambda \cdot x(s)ds + 2\int_0^t |x^X(s)|^2 ds + 2x(t) \cdot Y_0\right)}$$

Implication I

$$\dot{x} = Ax + B \begin{pmatrix} x_1^2 \\ \dots \\ x_{n+m}^2 \end{pmatrix}$$

1. $x(0) = 0 \Longrightarrow x(t) \equiv 0$: origin is an equilibrium

2. Jacobian at the origin $J(0) = A$: negative eigenvalues
   $\Longrightarrow \lim_t x(t) = 0$ whenever $x(0) \in U$ some neighborhood of the origin

3. For any u and t, $\mathbb{E}\exp(\theta u \cdot Y_t) < \infty$ for small θ
   $\Longrightarrow u \cdot Y_t$ cannot have heavy tails

Implication II

$$\boxed{S_t := \{u : |x(t)| < \infty\}}$$

$u \cdot Y_t$ has light left and light right tails

$$\Updownarrow$$

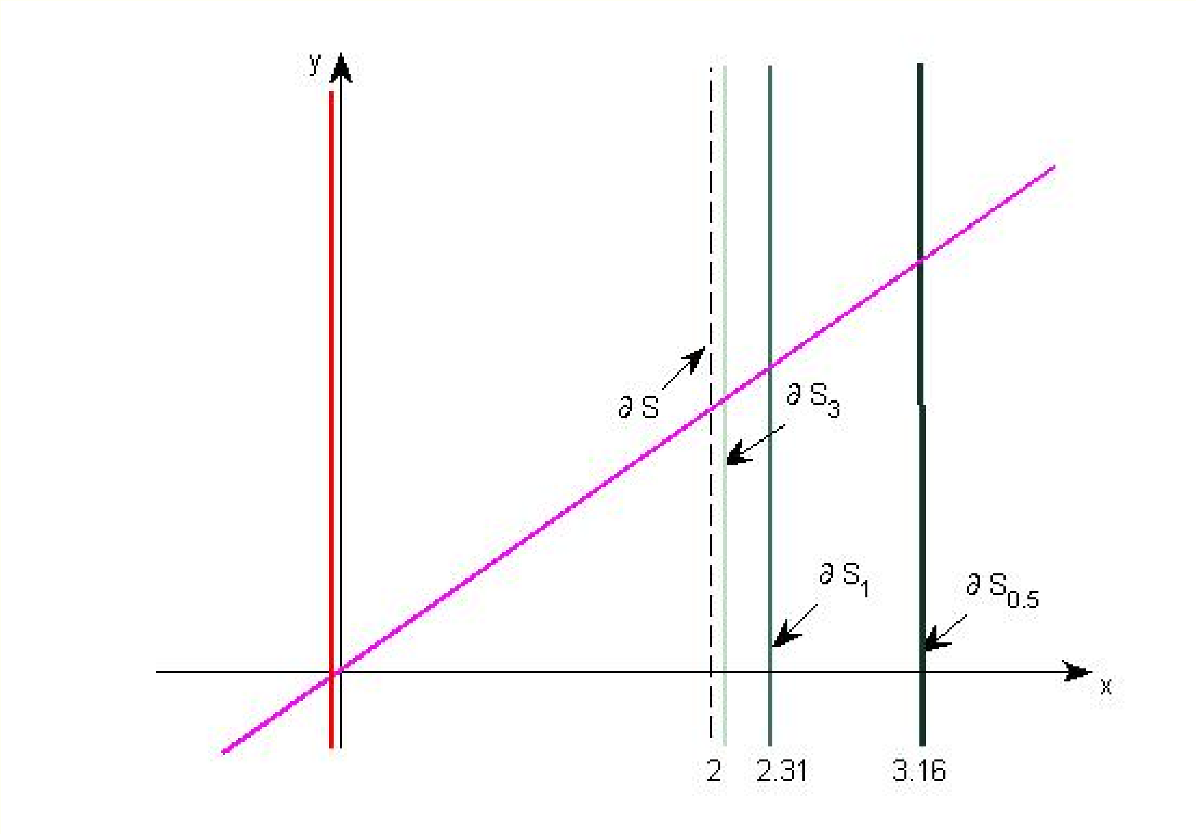
$\mathbb{E}\exp(\theta u \cdot Y_t) < \infty$ for all $\theta \in \mathbb{R}$

$$\Updownarrow$$

$\theta u/2 \in S_t$ for all $\theta \in \mathbb{R}$

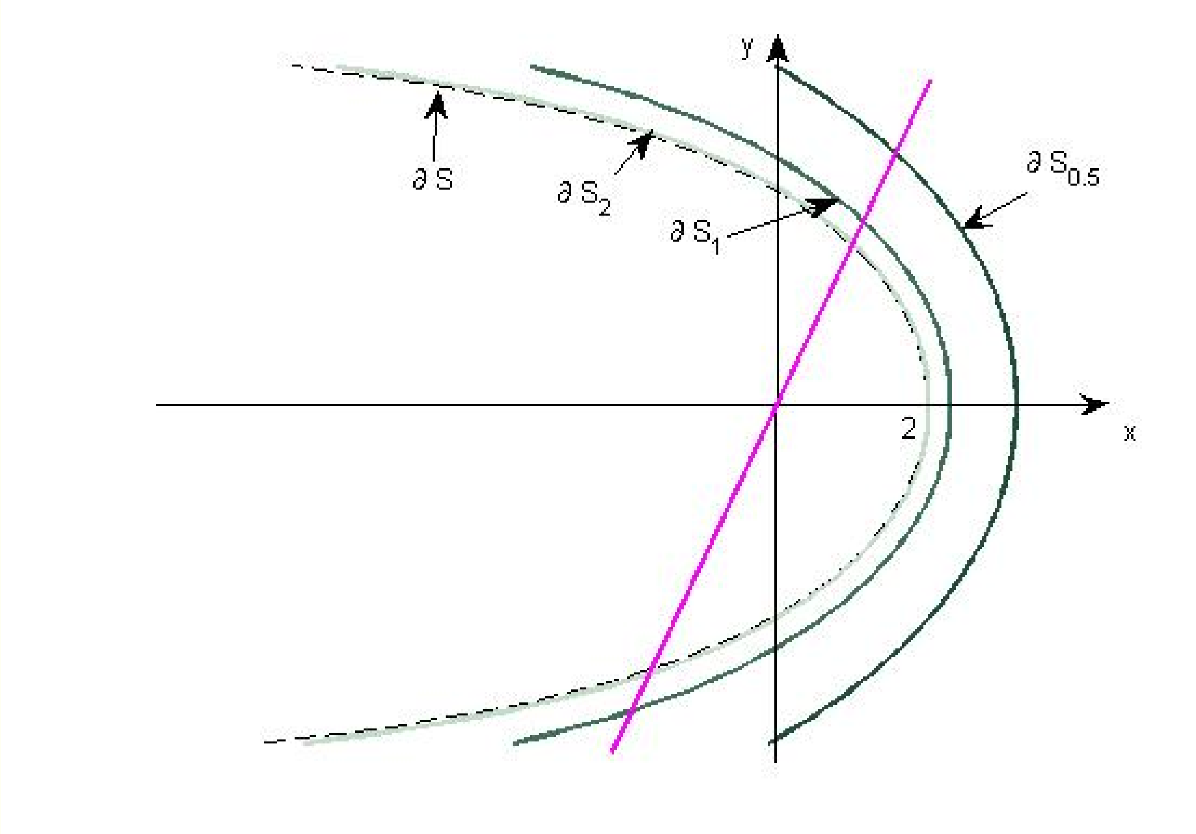
(CIR, OU)	Heston-like
$dV_t = (m_1 - 2V_t)dt + \sqrt{V_t}dW_t^1$	$dV_t = (m_1 - 2V_t)dt + \sqrt{V_t}dW_t^1$
$dX_t = (m_2 - 2X_t)dt + dW_t^2$	$dX_t = (m_2 - 2X_t)dt + \sqrt{1 + V_t}dW_t^2$

$Y_t =$(CIR, OU), $S_t := \{u : |x(t)| < \infty\}$



- $u = (1,0) \Longrightarrow u \cdot Y_t =$CIR
- $u = (0,1) \Longrightarrow u \cdot Y_t =$OU
- $u \cdot Y_t$ has light tails if and only if $u_1 = 0$

$Y_t =$Heston-like, $S_t := \{u : |x(t)| < \infty\}$



- For any u, $u \cdot Y_t$ has exponentially bounded tails

Stationary Distribution

Examples

1. Vasicek: $dY(t) = \alpha(b - Y(t))dt + \sigma dW(t)$, $Y/\sigma \in \mathbb{A}_0(1)$

$$Y(t) \sim N\left(e^{-\alpha t}(Y_0 - b) + b, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right)$$

$$Y_\infty \sim N\left(b, \frac{\sigma^2}{2\alpha}\right)$$

2. Cox-Ingersoll-Ross: $dY(t) = \alpha(b - Y(t))dt + \sigma\sqrt{Y(t)}dW(t)$, $Y/\sigma^2 \in A_1(1)$

$$Y(t) \sim \frac{\sigma^2(1 - e^{-\alpha t})}{4\alpha}\chi_d'^2\left(\frac{4\alpha e^{-\alpha t}}{\sigma^2(1 - e^{-\alpha t})}Y(0)\right), \quad d = \frac{4b\alpha}{\sigma^2}$$

$$Y_\infty \sim \frac{\sigma^2}{4\alpha}\chi_d^2, \quad \mathbb{E}e^{2uY_\infty} = \left(1 - \frac{u}{\alpha}\right)^{-d/2}, \quad u < \alpha$$

Theorem 2

$\exists!$ limiting stationary distribution Y_∞ for any $Y_0 \in \mathbb{R}_+^m \times \mathbb{R}^n$.

$$\boxed{\mathbb{E}e^{u \cdot Y_\infty} = \exp\left(2\int_0^\infty \Lambda \cdot x(t)dt + 2\int_0^\infty |x^d(t)|^2 dt\right)}$$

for $u \in = \{u : \mathbb{E}e^{u \cdot Y_\infty} < \infty\}$. Moreover,

$$S = \{u : \lim_{t \to \infty} x(t) = 0, x(0) = u\} = \{u/2 : \mathbb{E}e^{u \cdot Y_\infty} < \infty\}$$

$$\dot{x} = Ax + B\left(x_1^2, \ldots, x_{m+n}^2\right), \quad x(0) = u/2$$

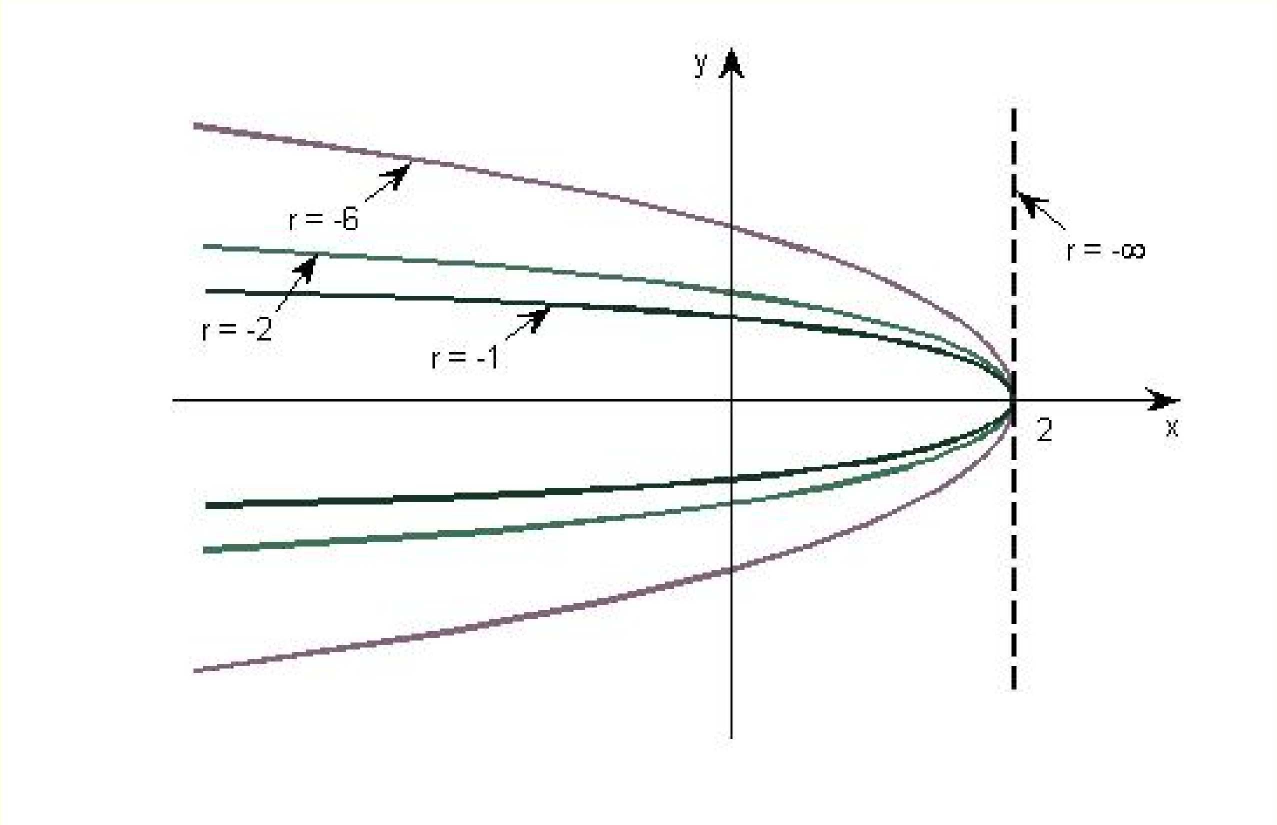
Implication

1. Stability region: open set containing the origin $\Rightarrow u \cdot Y_\infty$ never achieves a heavy tail

2. $u \cdot Y_\infty$ has a light tail iff $\theta u \in S$ for all $\theta \in \mathbb{R}$

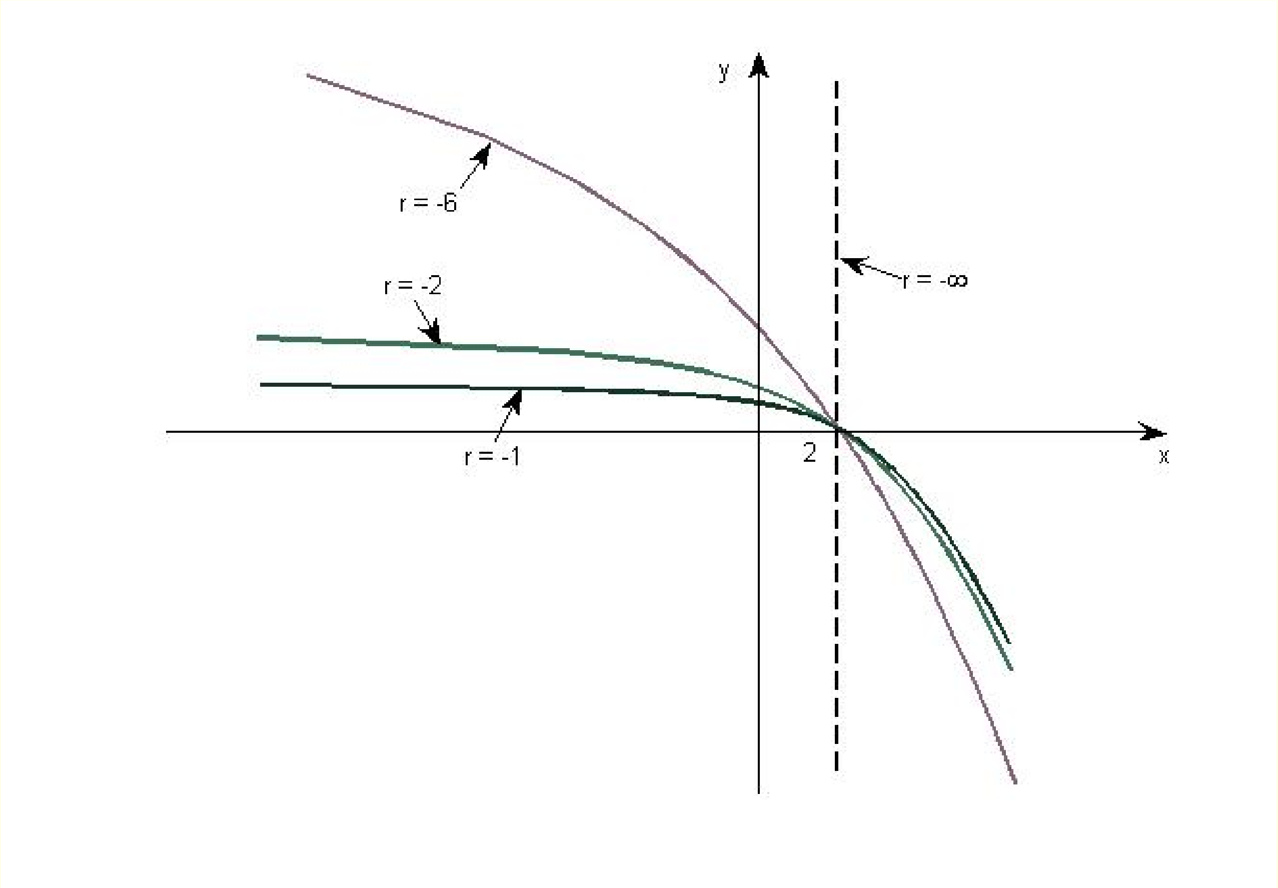
Example I $\mathbb{A}_1(2)$

$$\begin{aligned} dV_t &= (m_1 - 2V_t)dt + \sqrt{V_t}dW_t^1 \\ dX_t &= (m_2 + rX_t)dt + \sqrt{1+V_t}dW_t^2 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \dot{x}_1 &= -2x_1 + x_1^2 + x_2^2 \\ \dot{x}_2 &= rx_2 \end{aligned}$$



Example II $\mathbb{A}_1(2)$

$$\begin{aligned} dV_t &= (m_1 - 2V_t)dt + \sqrt{V_t}dW^1 \\ dX_t &= (m_2 + V_t + rX_t)dt + dW^2 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \dot{x}_1 &= -2x_1 + x_2 + x_1^2 \\ \dot{x}_2 &= rx_2 \end{aligned}$$



Gaussian Conditions

Jordan Canonical Form of $-K_2^\top$

$$dX_t = (K_2(\Theta_2 - X_t) - K_3 V_t)\,dt + \sqrt{diag(1 + MV_t)}dW_t^2$$

$\lambda_1, \ldots, \lambda_k$: distinct eigenvalues of $-K_2^\top$
$\Rightarrow \exists\ P$ s.t. $P^{-1}(-K_2^\top)P = J$ where

$$J = \left(\begin{array}{c|c|c} J_1 & & \\ \hline & J_2 & \\ \hline & & J_k \end{array}\right), \quad J_i = \left(\begin{array}{c|c|c} J_i^1 & & \\ \hline & J_i^2 & \\ \hline & & J_i^{g_{\lambda_i}} \end{array}\right)$$

J_i^j: Jordan blocks associated with λ_i

g_{λ_i}: geometric multiplicity $= \dim \mathrm{Ker}(-K_2^\top - \lambda_i I) =$ the number of Jordan blocks for λ_i

a_{λ_i}: algebraic multiplicity $= \sum_j \dim J_i^j = \dim J_i$

Useful Matrices

Define $N_i = J_i - \lambda_i I$, $\tilde{u} = P^{-1} u^X$

$$W := \begin{pmatrix} w_1 \\ \vdots \\ w_l \\ \hline \\ -K_3^\top P \\ \end{pmatrix} = \left[\begin{array}{c|c|c} W_1 & \cdots & W_k \end{array}\right], \quad \tilde{u} = \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^k \end{pmatrix}, \quad \tilde{u}^i \in \mathbb{R}^{a_{\lambda_i}}.$$

$w_1, \ldots, w_l$: q-th rows of P for $q \in \mathcal{J} = \{q : \exists p \text{ s.t. } M_{pq}^\top \neq 0\}$

W_1: the first a_{λ_1} columns of W, W_2: the next a_{λ_2} columns, etc.

Theorem 3

For any given $t > 0$ and $u \in \mathbb{R}^{m+n}$,

$$\mathbb{E}e^{\theta u \cdot Y_t} < \infty \text{ for all } \theta \in \mathbb{R}$$

if and only if $u^V = 0$ and

$$W_i N_i^l \tilde{u}^i = 0, \quad l = 0, \ldots, a_{\lambda_i} - 1, \quad i = 1, \ldots, k$$

Moreover, $u \cdot Y_t$ has a Gaussian distribution if and only if these conditions hold.

Remark

To achieve a light tail (in this case, Gaussian), we remove all the dependence on V factors from $u \cdot Y_t = u^V \cdot V_t + \tilde{u}^X \cdot (P^\top X_t)$ and this dependence is captured by the above conditions.

Example $\mathbb{A}_1(3)$

$$
\begin{aligned}
dY_1 &= (m_1 - Y_1)dt + \sqrt{Y_1}dW_t^1 \\
dY_2 &= (m_2 + aY_1 - Y_2)dt + dW_t^2 \\
dY_3 &= (m_3 + bY_1 + cY_2 - Y_3)dt + dW_t^3
\end{aligned}
$$

For no explicit dependence on Y_1, $u = (0, u_2, u_3)$. Then,

$$d(u \cdot Y) = (\Lambda + (au_2 + bu_3)Y_1 + cu_3Y_2 - u \cdot Y)dt + u_2 dW_t^2 + u_3 dW_t^3$$

- $au_2 + bu_3 = 0$ (*)
- If $a \neq 0$, then $cu_3 = 0$
- If $a \neq 0$ and $c = 0$, then $u \cdot Y_t$ is Gaussian under (*)
- If $a \neq 0$ and $c \neq 0$, then $u_2 = u_3 = 0$: no $u \cdot Y$ is Gaussian

Example $\mathbb{A}_1(4)$

$$-K_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

$$\begin{aligned} d\left(u^X \cdot X_t\right) &= \left(u^X \cdot (K_2\Theta_2) - \sum_j u_j^X (K_3)_j V_t + \sum_j u_j^X \lambda_j X_{t,j}\right) dt \\ &\quad + \sum_j u_j^X \sqrt{1 + M_j V_t} dW_j^2(t) \end{aligned}$$

$\Rightarrow$ For no (explicit) dependence on V,

(1) $u^V = 0$,

(2) $\sum_j u_j^X (K_3)_j = 0$

(3) $u_j^X = 0$ for all $j \in \mathcal{J} = \{j : M_j \neq 0\}$.

However, this is not enough.

$$\begin{aligned} d\left(u^X \cdot X_t\right) &= \left(u^X \cdot (K_2\Theta_2) + \lambda_1\left(u^X \cdot X_t\right) + (\lambda_2 - \lambda_1)u_3^X X_{t,3}\right) dt \\ &\quad + \sum_{j \notin \mathcal{J}} u_j^X dW_j^2(t) \end{aligned}$$

If X_3 depends on V, then $u \cdot Y$ is not free of $V \Rightarrow u_3^X = 0$:

$$d\left(u^X \cdot X_t\right) = \left(u^X \cdot (K_2\Theta_2) + \lambda_1\left(u^X \cdot X_t\right)\right) dt + \sum_{j \notin \mathcal{J}} u_j^X dW_j^2(t)$$

Conclusion

Summary

- The transform formula as in DPS holds unconditionally for the canonical affine diffusion models
- $u \cdot Y_t$ has necessarily tails with exponential decay or Gaussian distribution and this Gaussian case is achieved when u satisfies certain constraints
- Canonical models admit limiting stationary distributions Y_∞ and they are characterized by MGFs, which in turn are expressed using solutions to Riccati ODEs
- The region of finite exponential moments of Y_∞ coincides with the stability region of the associated ODEs
- The stability region S of the origin shows the tail behaviors of $u \cdot Y_\infty$ at least qualitatively

Thank You!