Calibration of portfolio credit derivative models : solution of an inverse problem via intensity control

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- A general parametrization of the portfolio loss process
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Portfolio credit derivatives

Contracts whose payoffs depend on the losses due to defaults in some underlying reference portfolio (of loans, bonds or credit default swaps).

Most common example: Collateralized Debt Obligations (CDOs).

Commonly used approach to pricing of portfolio credit derivatives: value = discounted expectation of cash flows computed under a *pricing measure* ("risk neutral probability") $\mathbb{Q}$:

$$V_t = \sum_{t_j > t} E^{\mathbb{Q}}[B(t, t_j) H_j(L(t_j))] \tag{1}$$

where t_j are cash flow dates, $L(t_j)$ is the loss due to default in the reference portfolio, $B(t, t_j)$ is the discount factor and $H_j(L(t_j))$ is the random, default dependent cash flow paid at t_j.

Example: Channel CDO Ltd

Structurer: PIMCO. Pool: 37.5 Billion$ of CDS and corporate bonds.

Rated by Moody's

Tranche	Amount	Percent	Rating	Spread
Supersenior	840	82.76	–	14.5 bp
Class A	90	8.87	Aaa	LIBOR+ 50 bp
Class B	20	1.97	Aa3	LIBOR+100 bp
Class C	15	1.48	A3	LIBOR+ 225
Class D	12.5	1.23	Baa2	LIBOR+325
Equity	37.5	3.69	–	—-

Ingredients

- $i = 1..n$ obligors
- Total nominal nA
- Default dates $T_i, i = 1..n$
- Number of defaults in $[0, t]$: N_t
- Portfolio loss (as percentage of total nominal): $L_t = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) 1_{T_i \leq t}$
- Tranche loss: $L_{a,b}(t) = (L(t) - a)^+ - (L(t) - b)^+$

Cash flow structure of a CDO tranche

Default leg: tranche loss due to defaults between t_{j-1} and t_j

$$\begin{aligned}
\text{Cash flow at } t_j \quad & [L_{a,b}(t_j) - L_{a,b}(t_{j-1})] \\
\text{Value at } t = 0 \quad & \sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ ] \qquad (2) \\
= \quad & \sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[(L(t_j) - a)^+ - (L(t_j) - b)^+ \\
& \quad -(L(t_{j-1}) - a)^+ \ + (L(t_{j-1}) - b)^+]
\end{aligned}$$

Similar to pricing of a portfolio of calls on $L(t)$.

Requires knowledge of the risk neutral distribution of total portfolio loss $L(t)$

Premium leg: pays fixed spread S(a,b) at dates t_j on remaining principal

Cash flow at t_j	$S(a,b)(t_j - t_{j-1})[(b - L(t_j))^+ - (a - L(t_j))^+]$
Value at $t = 0$	$S(a,b)\sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1})$
	$E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+]$

Computation of $E^{\mathbb{Q}}[(L(t_j) - K)^+]$ requires knowledge of the (risk neutral) distribution of total loss $L(t_j)$ which depends on dependence among defaults.

Fair spread of a CDO tranche swap with attachment point a and detachment b initiated at $t = 0$:

$$S_0(a,b) = \frac{\sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ ]}{\sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1}) E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+]}$$

Computation of CDO spread involves $E^{\mathbb{Q}}[(L(t_j) - K)^+]$ which requires knowledge of the (risk neutral) distribution of total loss $L(t_j)$: involves assumptions on dependence among defaults ("default correlation")

Mark to market value of the value of a protection seller on a nominal X of the tranche: premium leg- default leg

$$MTM(t) = XS_0(a,b)\sum_{t_j>t} B(t,t_j)\delta_j E^{\mathbb{Q}}[(b-L(t_j))^+ - (a-L(t_j))^+|\mathcal{F}_t]$$

$$-X\sum_{t_j>t} B(t,t_j)E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})|\mathcal{F}_t]$$

$$= X(b-a)\sum_{t_j>t} B(t,t_j)[S_0(a,b)\delta_j(1-P_{a,b}(t,t_j))-$$

$$= [S_0(a,b) - S_t(a,b)]\ X\sum_{t_j>t} B(t,t_j)\delta_j E^{\mathbb{Q}}[(b-L(t_j))^+ - (a-L(t_j))^+|\mathcal{F}_t]$$

where $\delta_j = t_j - t_{j-1}$.

Case of the equity tranche $[0, K]$

Default leg: tranche loss due to defaults between t_{j-1} and t_j

Cash flow at t_j $\quad [\min(L(t_j), K) - \min(L(t_{j-1}), K)]$

Value at $t = 0$ $\quad \sum_{j=1}^{J} B(0, t_j) E^{\mathbb{Q}}[\min(L(t_j), K) - \min(L(t_{j-1}), K)\ ]$

Premium leg: upfront fee $U(K)\%$ of the nominal of the tranche+ fixed spread f (usually 500 bp) at dates t_j on remaining principal

Cash flow at t_j $\qquad f(t_j - t_{j-1})(K - L(t_j))^+$

Value at $t = 0$ $\qquad f \sum_{j=1}^{J} B(0, t_j)(t_j - t_{j-1}) E^{\mathbb{Q}}[(K - L(t_j))^+]$

$+KU(K)$

Upfront fee for equity tranche with detachment point K:

$$KU(K) = \sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[\min(L(t_j), K) - \min(L(t_{j-1}, K)\ ]$$
$$-f \sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1}) E^{\mathbb{Q}}[(K - L(t_j))^+]$$

Computation requires knowledge of the (conditional) distribution $P_x(t,t_j) = \mathbb{Q}(L(t_j) \leq x | \mathcal{F}_t)$ of total loss $L(t_j)$ which depends on dependence among defaults

Index default swap

Protection seller agrees to pay all default losses in the index in return for a fixed spread on notional of *remaining obligors*:

Default leg: pays tranche loss due to defaults between t_{j-1} and t_j

$$\text{Cash flow at } t_j \qquad [L(t_j) - L(t_{j-1})]$$

$$\text{Value at } t = 0 \qquad \sum_{j=1}^{J} E^{\mathbb{Q}}[B(0,t_j)L(t_j) - L(t_{j-1})\ ]$$

Premium leg: spread S paid on remaining obligors

$$\text{Cash flow at } t_j \qquad (1 - \frac{N(t_j)}{n})\ S(t_j - t_{j-1})$$

$$\text{Value at } t = 0 \qquad S\sum_{j=1}^{J} E^{\mathbb{Q}}[B(0,t_j)(1 - \frac{N(t_j)}{n})(t_j - t_{j-1})\ ]$$

Index default swap spread (for an equally weighted index)

Index spread at $t = 0$: equalizes two legs at inception

$$S_{\text{index}}(0) = \frac{E^{\mathbb{Q}}[\sum_{j=1}^{J} B(0,t_j)L(t_j) - L(t_{j-1})\ ]}{E^{\mathbb{Q}}[\sum_{j=1}^{J} B(0,t_j)(1 - \frac{N(t_j)}{n})(t_j - t_{j-1})\ ]} \tag{3}$$

Depends on joint law of loss value L_t and number of defaults N_t

In the case where recovery rates are constant and equal, $L_t = \frac{(1-R)N_t}{n}N$ so the index default spread only depends on the law of $L(t_j)$ at payment dates.

The spread of a CDO tranche is given by

$$S_t(a,b) = \frac{\sum_{j=1}^m B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1}) \ |\mathcal{F}_t]}{\sum_{j=1}^m B(0,t_j)\delta_j E^{\mathbb{Q}}[(b-L(t_j))^+ - (a-L(t_j))^+|\mathcal{F}_t]}$$

Key observation: these expressions for the tranche spreads depend on the portfolio loss process via the expected tranche notional

$$C(t_j, K) = E^{\mathbb{Q}}[(K - L_{t_j})^+]. \tag{4}$$

Given tranche spreads simple algorithms can be used to "strip" a set $C(t_j, K_i), j = 1..J, i = 1..I$ of expected tranche notionals.

See e.g.: C & Savescu 2006, Arnsdorff & Halperin 2007,...

Aggregate loss modeling

Idea: model portfolio loss up to t as a jump process L_t with increasing, piecewise constant sample paths whose jump times T_j are the default events and whose jump sizes L_j are default losses:

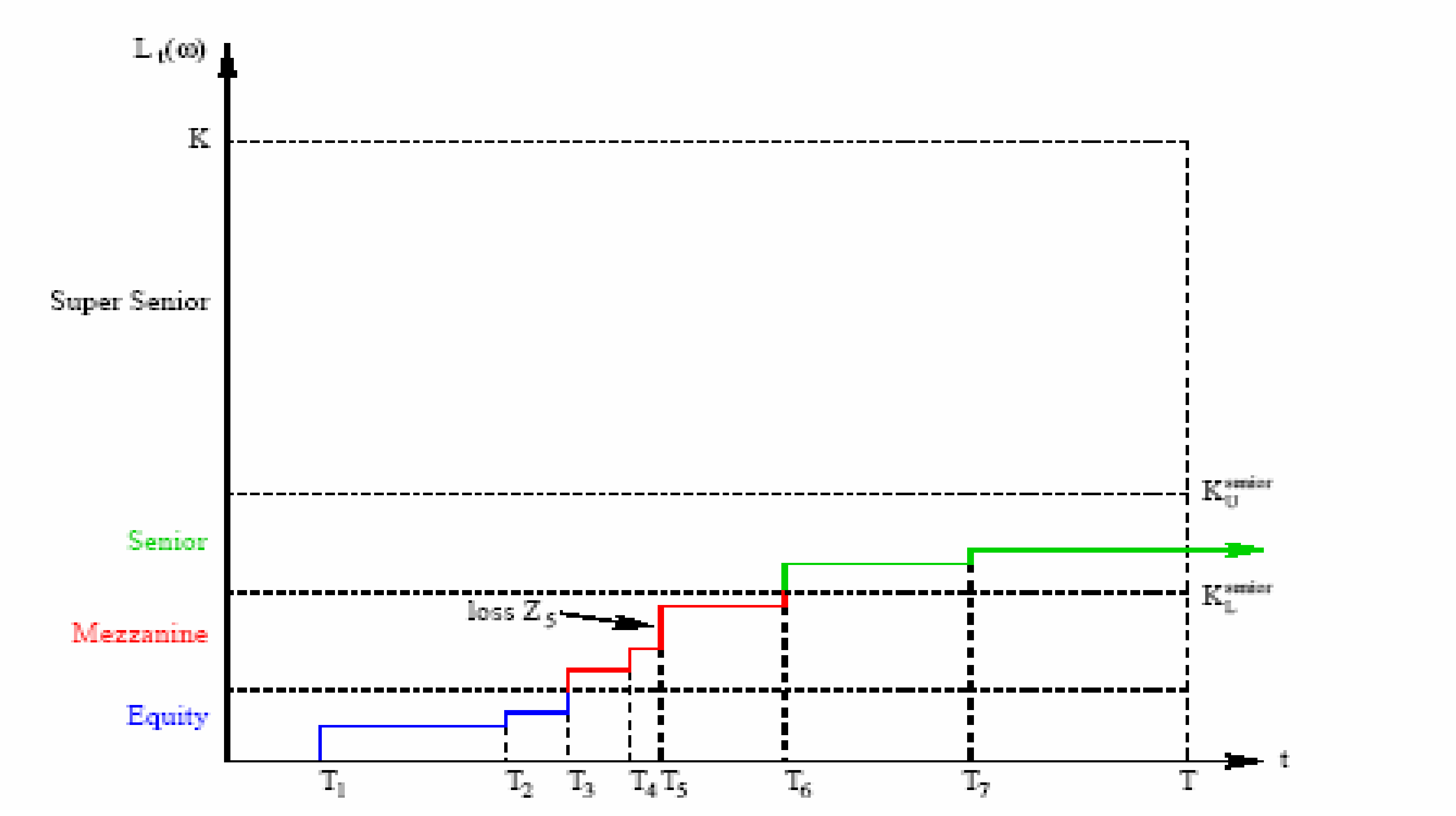
$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)1_{\tau_i \le t} = \sum_{j=1}^{N_t} L_j \qquad (5)$$

where $N_t = \sum_{i=1}^{n} 1_{\tau_i \le t}$ is the number of defaults in portfolio before t and L_j is loss at j-th default event.

Key ingredient: *aggregate default rate* λ_t defined as probability per unit time of the next default conditional on current market information

$$\mathbb{Q}[N_{t+\Delta t} = N_t + 1 | \mathcal{F}_t] = \lambda_t \Delta t + o(\Delta t)$$

Sample path of the loss process



- Arrival rate λ_t of (next) default event:

$$\lambda_t(\omega) = \lim_{\Delta t \to 0} \frac{\mathbb{Q}(N(t+\Delta t) = N(t-) + 1 | \mathcal{F}_t)}{\Delta t}$$

  – Poisson: $\lambda_t = f(t)$

  – "Doubly stochastic": default intensity driven by other "market factors", not by default itself

$$d\lambda_t = \mu(t, \lambda_t)dt + \sigma(t, \lambda_t)dW_t$$

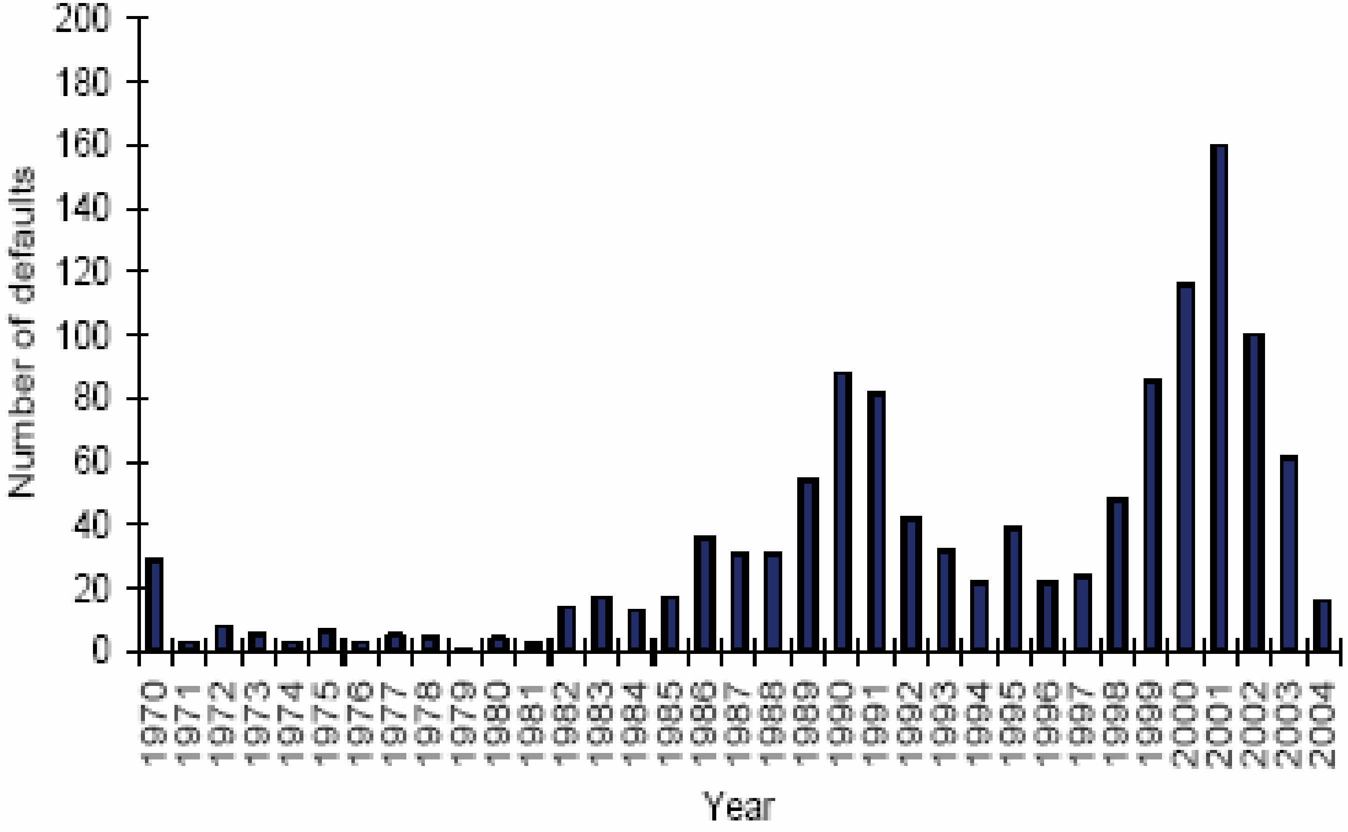
  – Inhomogeneous Markov process: $\lambda_t = f(t, N_t) = a_{N_t}(t)$ where $a_n(t)$ are transition rates from n to $n+1$

  – Dependence on history of defaults/ losses:

$$\lambda_t = g(\tau_j, L_j, j = 1..N_t - 1)$$

- Distribution of loss given default L_j: typically IID with some law F

Clustering of defaults



Common approach in this literature: specify a parametric model in one of these classes and estimate coefficients by fitting the data to the model by nonlinear least squared via a brute force optimization.

Many possible choices for modeling L_t: compound Poisson process (Brigo & Pallavicini 05), self-exciting point process (Giesecke & Goldberg), Cox process (Longstaff & Rajan), conditional Markov chain (Ehlers & Schonbucher 07).

Which one to pick? how to choose its parameters (loss intensity) consistently with market observations of CDO spreads?

Information content of credit portfolio derivatives

Market observations at $t = 0$ consist of fair spreads for (index) CDO tranches. These can be equivalently represented in terms of expected tranche notionals at payment dates:

$$C(t_j, K) = E^{\mathbb{Q}}[(K - L_{t_j})^+ | \mathcal{F}_0]. \quad (6)$$

These quantities only depend on the process L via the *distribution* of L_{t_j} at the payment dates t_j.

How complex does a model need to be to reproduce/fit/model such quantities?

Mimicking arbitrary jump processes with Markovian ones

Proposition 1. *Consider any (non-explosive) jump process* (L_t) *defined by a jump intensity* $(\lambda_t(\omega))_{t\in[0,T^*]}$ *and IID jump sizes with distribution* F*. Then there exists a Markovian jump process* $\tilde{L}_T$ *with same jump size distribution and jump intensity given by:*

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0] \tag{7}$$

such that, conditional on $\mathcal{F}_0$, L_t *and* $\tilde{L}_t$ *have the same distribution for each* t*. In particular, the flow of marginal distributions of* L_t *only depends on* $(\lambda_t)_{t\in[0,T^*]}$ *through the effective intensity* $\lambda_{\text{eff}}(.,.)$.

Remark: this is a discontinuous version of Gyöngy's (1990) theorem for Brownian martingales.

Proof. Consider any bounded measurable function $f(.)$. Decomposing the loss into the sum of its jumps yields

$$\begin{aligned}
f(L_T) &= f(L_0) + \sum_{0<s\leq T} (f(L_{s-} + \Delta L_s) - f(L_{s-})\ ) \quad \text{so} \\
E[f(L_T)|\mathcal{F}_0] &= f(L_0) + E[\sum_{0<s\leq T} (f(L_{s-} + \Delta L_s) - f(L_{s-})\ )|\mathcal{F}_0] \\
&= f(L_0) + \int_0^T dt \quad E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{F}_0]
\end{aligned}$$

Denote $\mathcal{G}_t = \sigma(\mathcal{F}_0 \vee L_{t-})$ the information set obtained by adding the knowledge of running loss L_{t-} to the current information set $\mathcal{F}_0$. Define the *local intensity* function as

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t|\mathcal{F}_0, L_{t-} = l] = E^{\mathbb{Q}}[\lambda_t|\mathcal{G}_t] \tag{8}$$

We can now construct (the law of) a Markovian loss process $(\tilde{L}_t)_{0\leq t\leq T}$ with intensity $\gamma_t = \lambda_{\text{eff}}(t,\tilde{L}_{t-})$ and jump size distribution F.

Noting that $\mathcal{F}_0 \subset \mathcal{G}_t$ we have

$$E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{F}_0]$$

$$= E[E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{G}_t]|\mathcal{F}_0]$$

$$= E[\int_0^1 F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ E[\lambda_0|\mathcal{G}_t]|\mathcal{F}_0]$$

$$= E[\lambda_{\text{eff}}(t, L_{t-}) \int F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ |\mathcal{F}_0] \qquad \text{so}$$

$$E[f(L_T)|\mathcal{F}_0] = f(L_t) +$$

$$E[\int_0^T dt\ \lambda_{\text{eff}}(t, L_{t-}) F \int F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ |\mathcal{F}_0]$$

The above equality shows that $E[f(L_T)|\mathcal{F}_t]$ has the same value as $E[f(\tilde{L}_T)|\mathcal{F}_0]$ where $(\tilde{L}_t)_{0 \le t \le T}$ which shows the result. $\square$

Corollary: The value $E[f(L_T)|\mathcal{F}_0]$ at $t = 0$ of any derivative whose payoff only depends on the aggregate loss L_T of the portfolio only depends on the transition rate $(\lambda_t)_{t\in[0,T^*]}$ through its risk-neutral conditional expectation with respect to the current loss level:

$$\lambda_{\text{eff}}(t, l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0] \tag{9}$$

In particular, CDO tranche spreads and CDO tranches values only depends on the transition rate $(\lambda_t)_{t\in[0,T^*]}$ through its conditional expectation $\lambda_{\text{eff}}(.,.)$.

Case of index default swap:

The cashflows of an *index default swap* actually depends on L_t *and* N_t. In the case where receovery rates are constant $L_t = a(1-R)N_t$ so the above results also applies to the index default swap, which only depends on

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0]$$

In the general case of random loss given default N_t is a path-dependent functional of L. We can analogously shown that the index default swap only depends on

$$\lambda_1(t,l,n) = E^{\mathbb{Q}}[\lambda_t | N_{t-} = n, L_{t-} = l, \mathcal{F}_0]$$

In the sequel we will consider the (usual) case of constant recovery where $L_t = a(1-R)N_t$.

Forward equation for expected tranche loss (Cont & Savescu 2006)

Using the fact that the effective intensity leads to a Markovian loss process we can derive a Dupire-type forward equation for the expected tranche loss $C(T,K) = E^{\mathbb{Q}^\lambda}[(K - L_T)^+]$

$$\frac{\partial C(T,K)}{\partial T} = -\lambda_{\text{eff}}(T, K - \delta K) C(T,K)$$
$$-(\lambda_{\text{eff}}(T, K - 2\delta K) - 2\lambda_{\text{eff}}(T, K - \delta K))C(T, K - \delta K)$$
$$-\sum_{i=1}^{k-2} (\lambda_{\text{eff}}(T, (i-1)\delta K) - 2\lambda_{\text{eff}}(T, i\delta K) + \lambda_{\text{eff}}(T, (i+1)\delta K))C(T,K)$$

where $\delta K = (1 - R)/n$ is the loss step.

Calibration by Relative entropy minimization under constraints

One period case: Buchen & Kelly, Avellaneda 1998

Diffusion models: Avellaneda Friedman Holmes Samperi 1997

Monte Carlo setting: Avellaneda et al 2001

Lévy processes: C. & Tankov 2004, 2006

Given market prices $C(T_i, K_i)$ of expected tranche notionals and a prior guess λ^0 for the loss intensity process, look for the loss process (parameterized by its default intensity $(\lambda_t)_{t\in[0,T^*]}$) "closest" in law the prior and matching the market observations of

$$\inf_{\lambda\in\Lambda} E^{\mathbb{Q}_0}\left[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\ln\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\right] \qquad \text{under} \qquad E^{\mathbb{Q}^\lambda}[\,(K_i - L_{T_i})_+\,] = C(T_i, K_i) \tag{10}$$

where $\mathbb{Q}^\lambda$ is the law of the point process with intensity process λ and $\mathbb{Q}_0$ is the law of the point process with intensity λ^0.

Using the previous result we can restrict Λ to *Markov jump processes* with time/state dependent intensity $\lambda(t, L_t)$.

Computation of entropy

Equivalent change of measure for point processes (Jacod 1980, Bremaud 1981)

Proposition 2. *Let N_t be a Poisson process on $(\Omega, \mathcal{F}_t, \mathbb{P})$. Let λ_t be an $\mathcal{F}_t$-predictable process such that*

$$\int_0^t \lambda_s ds < \infty \qquad \mathbb{P} - a.s.$$

Define the probability measure $\mathbb{Q}^\lambda$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T, \tag{11}$$

$$Z_t = \left(\prod_{T_n \le t} \lambda_{T_n} \right) exp \left\{ \int_0^t (1 - \lambda_s) ds \right\}$$

Then N_t is a point process with $\mathcal{F}_t$ intensity $(\lambda_t)_{t \in [0,T]}$ under $\mathbb{Q}^\lambda$.

Proposition 3 (Computation of relative entropy). *Denote by*

- $\mathbb{Q}_0$ *the law on* $[0,T]$ *of a standard unit intensity Poisson process and*
- $\mathbb{Q}^\lambda$ *the law on* $[0,T]$ *of the point process with intensity* $(\lambda_t)_{t\in[0,T]}$ *verifying* $\int_0^T \lambda_t dt < \infty$ $\mathbb{P}$*-a.s.*

The relative entropy of $\mathbb{Q}^\lambda$ *with respect to* $\mathbb{Q}_0$ *is given by:*

$$H(\mathbb{Q}^\lambda,\mathbb{Q}_0) = E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\ln\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] = E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_t \ln\lambda_t dt + T - \int_0^T \lambda_t dt] \quad (12)$$

We note in particular that

- $H(\mathbb{Q}^\lambda,\mathbb{Q}_0)$ is a (strictly) convex functional of (λ_t)
- $H(\mathbb{Q}^\lambda,\mathbb{Q}_0)$ is easily computable given a specification of the intensity process.

Dual problem

Define the Lagrangian

$$L_0(\lambda,\mu) = E^Q\Big(\sum_{\tau_i \le T} \ln \lambda_{\tau_i} + T - \int_0^T \lambda_s ds - \sum_{i=1}^{I} \mu_i((K_i - L_T)^+ - C_i)\Big)$$

Using convex duality arguments, the primal problem:

$$\inf_{\lambda\in\Lambda} E^{\mathbb{Q}_0}\Big[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} \ln \frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\Big] \quad \text{under} E^{\mathbb{Q}^\lambda}[(K_i - L_{T_i})_+] = C(K_i, T_i) \qquad (13)$$

is then equivalent to the dual problem

$$\sup_{\mu\in\mathbb{R}^I} \inf_{\lambda\in\Lambda} E^{\mathbb{Q}_\lambda}\Big(\sum_{\tau_i \le T} \ln \lambda_{\tau_i} + T - \int_0^T \lambda_s ds - \sum_{i=1}^{I} \mu_i((K_i - L_T)^+ - C_i)\Big) \qquad (14)$$

Intensity control problem

An *intensity control* problem is an optimization problem with a criterion of the type

$$J(\lambda) = E^{Q_\lambda}[\int_0^T c(s, \lambda_s, N_s)ds + \Phi_T(\lambda_T, N_T)],$$

where $c(s, \lambda_s, L_s)$ is a *running cost* and $\Phi_T(\lambda_T, L_T)$ represents the terminal cost.

Given that the objective function only depends on the flow of marginal distributions of N, Proposition 1 shows that for any optimal control λ^* there exists an optimal *feedback control* $\lambda_{\text{eff}}(t, N_t)$ such that $J(\lambda^*) = J(\lambda^*)$.

The solution of an intensity control problem can then be characterized in terms of a Hamilton Jacobi equation. (Boel & Vairya 1973, Bismut 1975, Bremaud 1981)

Hamilton Jacobi equations for intensity control

Proposition 4. *Assume there exists a bounded differentiable solution* $V : [0,T^*] \times N \to V(t,k)$ *such that for all* $k \in \mathbb{N}$

$$\frac{\partial V}{\partial t}(t,k) + \inf_{\lambda \in \Lambda_t} \lambda(t,k)[V(t,k+1) - V(t,k)] + c(t,\lambda(t,k),k) = 0, \quad (15)$$

$$V(T,k) = \Phi(T,k), \quad (16)$$

and for each $n \in N^+$, *a* $\mathcal{F}_t$*-predictable mapping* $t \to \lambda^*(t,N_t)$ *such that for each* $t \in [t_0,T]$

$$\lambda^*(t,k) = \underset{\lambda \in \Lambda_t}{argmin}\, \lambda(t,k)[V(t,k+1) - V(t,k)] + c(t,\lambda(t,k),k). \quad (17)$$

Then $\lambda_t^* = \lambda^*(t,N_t)$ *is an optimal control. Moreover* $V(t_0,N_{t_0}) = \inf_{\lambda \in \Lambda_t} E^{\mathbb{Q}_\lambda}[\int_{t_0}^T C(s,\lambda_s,N_s)ds + \Phi_T(\lambda)|\mathcal{F}_{t_0}]$.

In the case of a single maturity, our dual problem is an intensity control problem with running cost $(\ln \lambda(t, N_t) - 1)\lambda(t, N_t) + 1$ and terminal cost $\Phi(L_T) = (K_i - L_T)^+ - C_i$.

The Hamilton Jacobi equations are given by

$$\forall k = 1..n, \qquad \frac{\partial V}{\partial t}(t,k) + 1 - e^{-[V(t,k+1) - V(t,k)]} = 0 \tag{18}$$

$$V(T,k) = -\sum_{i=1}^{I} \mu_i((K_i - \frac{k(1-R)}{n})^+ - C_i) \tag{19}$$

This problem must be solved for each μ, and the result $J_\mu(\lambda^*) = V(0,0;\mu)$ optimized over μ to yield the optimal Lagrange multipliers μ^*. The intensity λ which calibrates the observed CDO tranche spreads is then given by the corresponding optimal control, which is explicitly computed as

$$\lambda^*(t,k) = e^{-[V(t,k+1;\mu^*) - V(t,k;\mu^*)]} \tag{20}$$

Proposition 5 (Solution of Hamilton-Jacobi equations). *The solution of the Hamilton-Jacobi system (18)–(19) can be represented as:*

$$V(t,k;\mu) = -\ln E^{\mathbb{Q}_0}[e^{-\sum_{i=1}^{I}\mu_i((K_i - N_T\frac{(1-R)}{n})^+ - C(K_i,T)\ )}|N_t = k] \quad (21)$$

Using an exponential change of variable, the Hamilton Jacobi system can be turned into a linear system of ODEs:

$$u(t,k;\mu) = e^{-V(t,k;\mu)} \qquad \text{solves} \qquad (22)$$

$$-\frac{\partial u(t,k;\mu)}{\partial t} + u(t,k;\mu) - u(t,k+1;\mu) = 0$$

which is recognized as the backward Kolmogorov equation associated to the Poisson process (prior), whose solution can be explicitly computed as above.

So our calibration problem becomes

$$\sup_{\mu \in \mathbb{R}^I} V(0,0;\mu) \qquad \text{where}$$

$$V(0,0,\mu) = T - \ln[\sum_{k=1}^{n} \frac{T^k}{k!} e^{-\sum_{i=1}^{I} \mu_i ((K_i - \frac{k(1-R)}{n})^+ - C(K_i,T)\ )}]$$

The last step is to solve this convex unconstrained minimization problem using a gradient-based algorithm.

Case of several maturities

In realistic cases where several maturities $0 \leq T_1 \leq T_2 \cdots \leq T_m$ are observed, the criterion to be optimized in the dual problem is of the form

$$J(\lambda) = E^Q[\int_0^{T_m} C(s,\lambda)ds + \Phi_{T_1}(N_{T_1}) + \Phi_{T_2}(N_{T_2}) + \ldots \Phi_{T_m}(N_{T_m})].$$

This can be treated as a sequence of optimal control problems in the following manner

$$\inf_{\lambda \in \Lambda} J(\lambda) = \inf_{\lambda \in \Lambda} E^{\mathbb{Q}_\lambda}[\int_0^{T_1} c(s,\lambda_s,N_s)ds + \Phi_{T_1}(N_{T_1}) + E[\Phi_{T_2}(N_{T_2}) + \ldots \Phi_{T_m}(N_{T_m})|\mathcal{F}_{T_1}]] =$$

$$\inf_{\lambda \in \Lambda} E^{\mathbb{Q}_\lambda}[\int_0^{T_1} c(s,\lambda_s,N_s)ds + \Phi_{T_1}(N_{T_1}) + \inf_{\lambda \in \Lambda_{T_1}} E[\Phi_{T_2}(N_{T_2}) + \ldots \Phi_{T_m}(N_{T_m})|\mathcal{F}_{T_1}]] \tag{23}$$

The last equality results from the dynamic programming principle.

$$E[\int_{T_1}^{T_m} \Phi_{T_2}(N_T + \ldots \Phi_{T_m}(N_{T_m}|\mathcal{F}_{T_1}] = E[\Phi_{T_2}(N_{T_2}) + \ldots \Phi_{T_m}(N_{T_m})|N_{T_1}]$$

If we take

$$\Theta(T_1, N_{T_1}) = \Phi_{T_1}(N_{T_1}) + \inf_{\lambda \in \Lambda_{T_1}} E[\int_{T_1}^{T_m} \Phi_{T_2}(N_{T_2}) + \ldots \Phi_{T_m}(N_{T_m})|N_{T_1}]$$

this problem reduces to the case of a single maturity, because Θ_{T_1} is $\mathcal{F}_{T_1}$-measurable.

Hence we have reduced the problem to a sequence of analytically solvable problems.

Reconstruction algorithm

1. Extract the expected tranche losses $C(t_i, K_j) = E^{\mathbb{Q}}[(K - L_T)^+]$ from the CDO tranche spreads.
2. Solve analytically the Hamilton-Jacobi equations (18)–(19) given $\mu \in \mathbb{R}^I$ to obtain $V(0, 0, \mu)$.
3. Optimize $V(0, 0, \mu)$ over $\mu \in \mathbb{R}^I$ using a gradient–based method:

$$\inf_{\mu \in \mathbb{R}^I} V(0, 0, \mu) = V(0, 0, \mu^*) = V^*(0, 0)$$

4. Given μ^* we can then compute the calibrated default intensity (optimal control) as follows:

$$\lambda^*(t, k) = e^{-[V(t,k+1,\mu^*) - V(t,k,\mu^*)]} \tag{24}$$

5. The calibrated default intensity $\lambda^*(.,.)$ can then be used to compute CDO spreads for different tranches, forward tranches

etc. in a straightforward manner: first we compute the expected tranche loss $C(T, K)$ by solving the forward equation

$$\frac{\partial C(T,K)}{\partial T} = -\lambda^*(T, K - \delta K)C(T,K)$$
$$-(\lambda^*(T, K - 2\delta K) - 2\lambda^*(T, K - \delta K))C(T, K - \delta K)$$
$$-\sum_{i=1}^{k-2} (\lambda^*(T,(i-1)\delta K) - 2\lambda^*(T, i\delta K) + \lambda^*(T,(i+1)\delta K))C(T,K)$$

6. Base correlation representation: In particular the calibrated default intensity can be used to “fill the gaps” in the base correlation surface in an arbitrage-free manner, by first computing the tranche spreads and then inverting the Gaussian copula model to obtain 'implied correlations'.

Empirical results: ITRAXX

Maturity	Low	High	Bid\ Upfront	Ask\ Upfront
5Y	0%	3%	11.75%	12.00%
	3%	6%	53.75	55.25
	6%	9%	14.00	15.50
	9%	12%	5.75	6.75
	12%	22%	2.13	2.88
	22%	100%	0.80	1.30
7Y	0%	3%	26.88%	27.13%
	3%	6%	130	132
	6%	9%	36.75	38.25
	9%	12%	16.50	18.00
	12%	22%	5.50	6.50
	22%	100%	2.40	2.90

Maturity	Low	High	Bid\ Upfront	Ask\ Upfront
10Y	0%	3%	41.88%	42.13%
	3%	6%	348	353
	6%	9%	93	95
	9%	12%	40	42
	12%	22%	13.25	14.25
	22%	100%	4.35	4.85

Table 1: ITRAXX tranche spreads, in bp. For the equity tranche the periodic spread is 500bp and figures represent upfront payments.

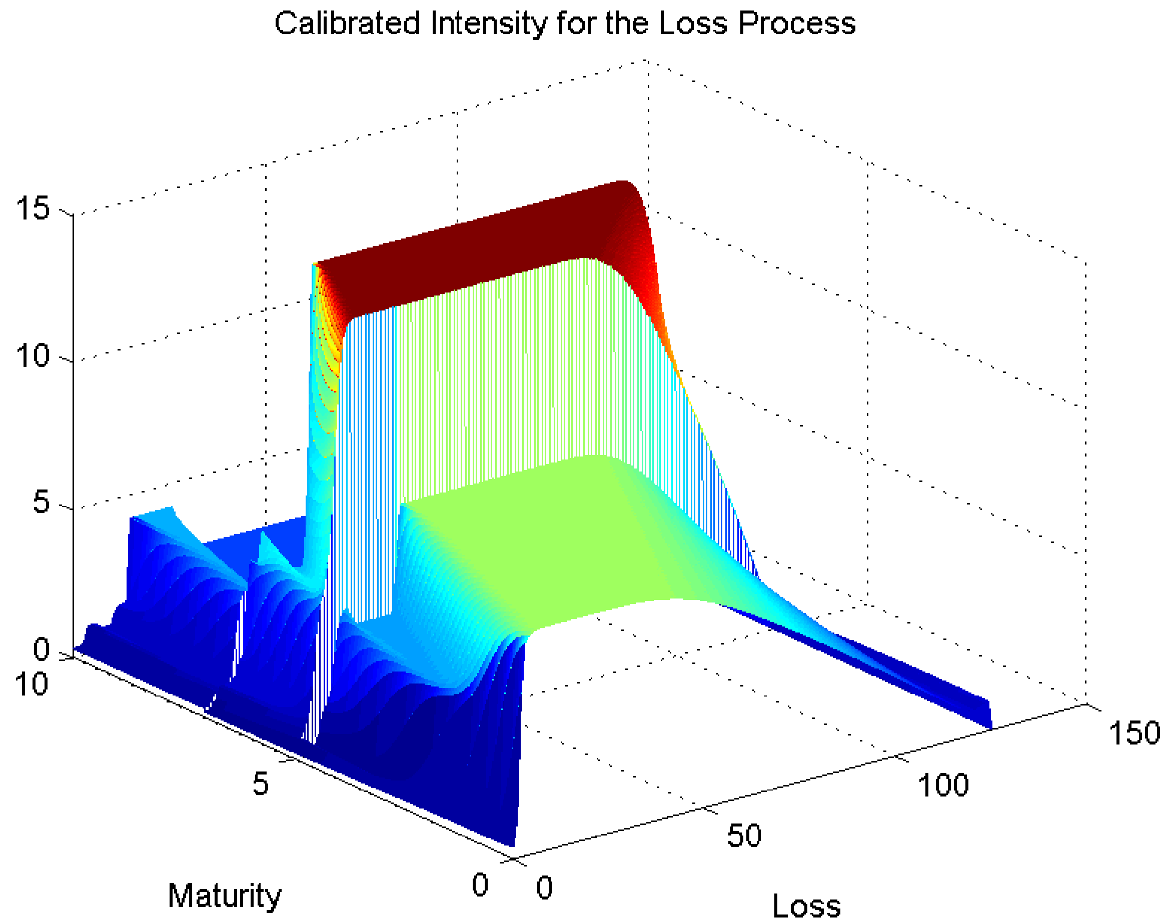


Figure 1: Calibrated intensity function $\lambda^*(t, L)$ for ITRAXX September 26, 2005.

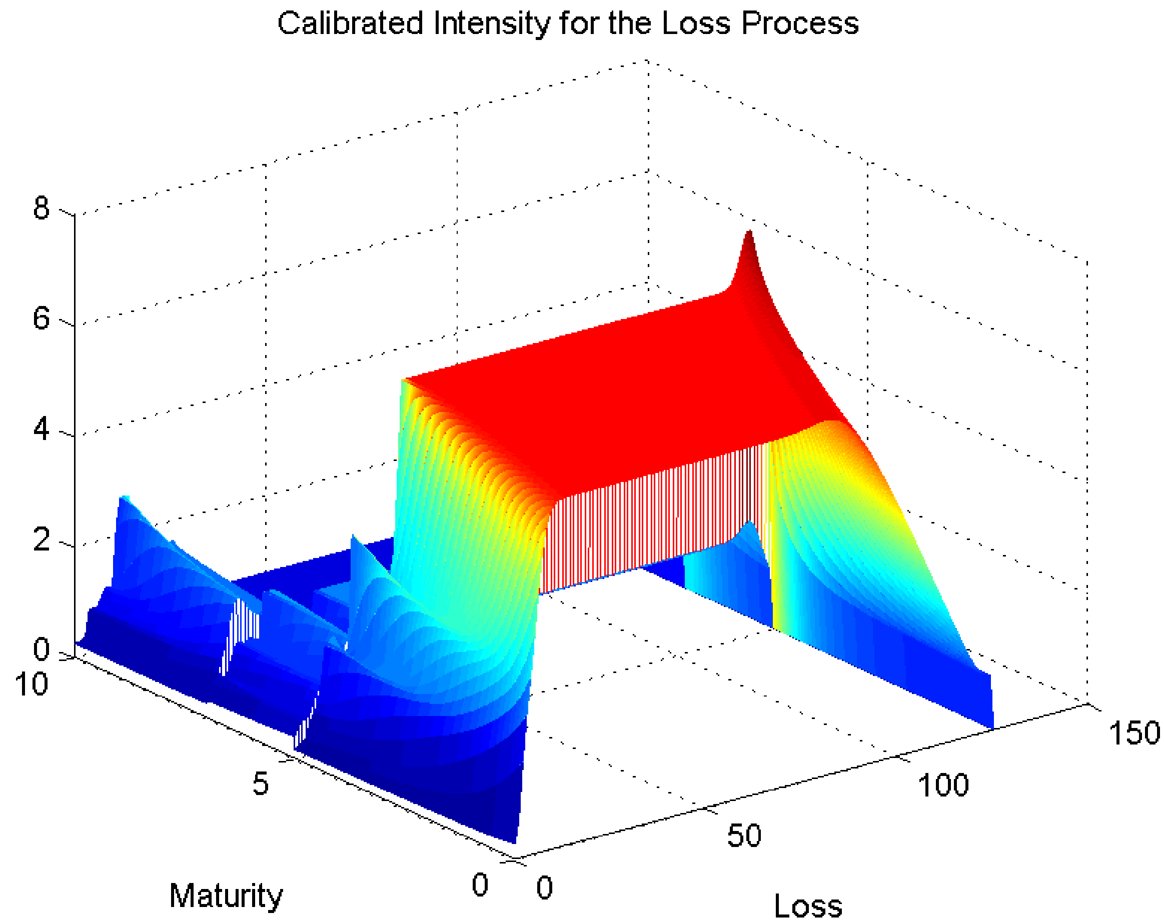


Figure 2: Calibrated intensity function $\lambda^*(t, L)$ for ITRAXX Europe Series 6, March 15 2007.

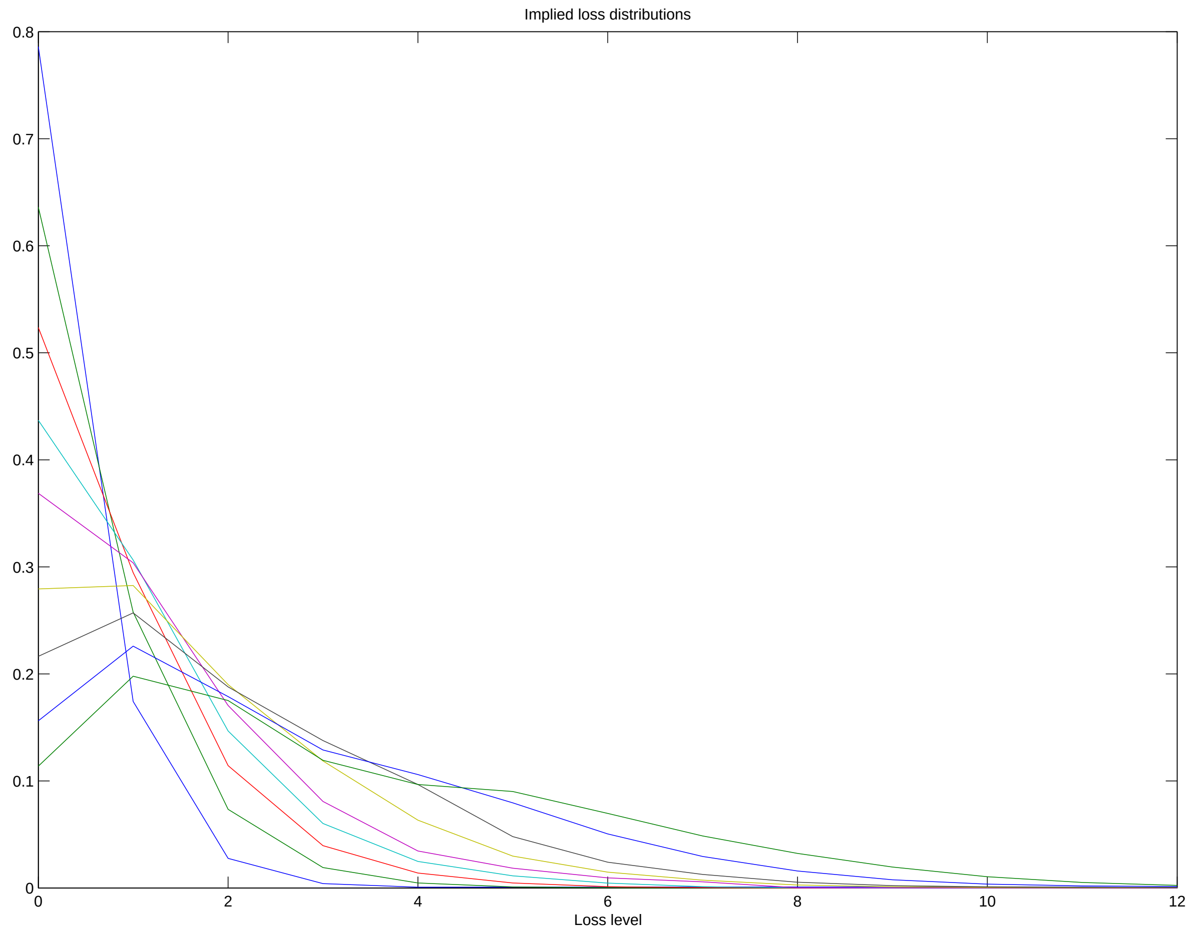


Figure 3: Implied loss distributions at various maturities: ITRAXX March 2007.

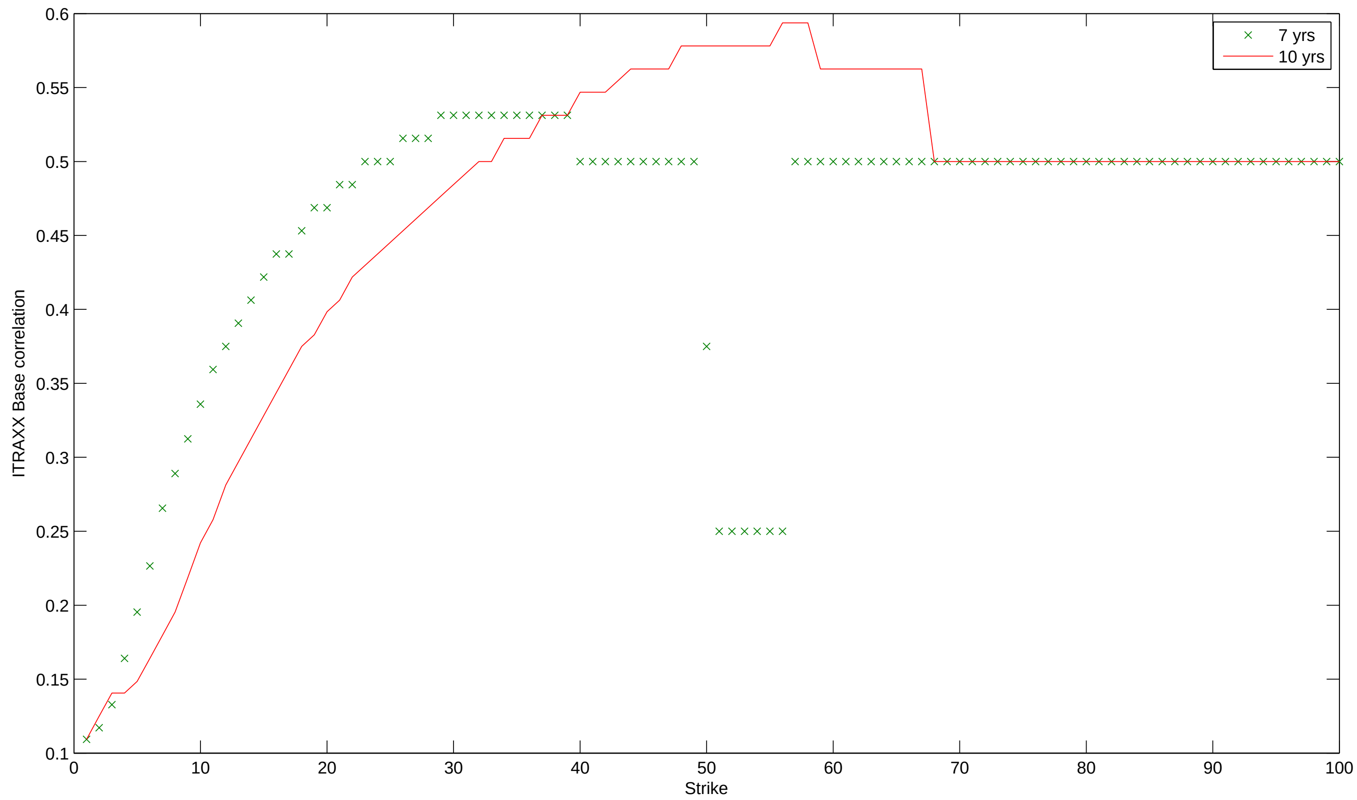


Figure 4: Base correlation surves generated by the calibrated model: ITRAXX.

Conclusion

- Stochastic control problems can be useful for solving model calibration problems.
- Rigorous methodology for calibrating a 'top-down' CDO pricing model to market data.
- Stable calibration algorithm based on intensity control method.
- No 'black box' optimization.
- Easy to implement/run: a few seconds on a laptop.
- Involves the solution of a first order linear system of ODEs+ unconstrained convex minimization in dimension $\simeq 20$.
- Results point to default contagion effects in the risk–neutral loss process.

Work in progress (with Yu Hang KAN, Columbia): hedge ratios for CDO tranches with index/other tranches, Forward tranche pricing