Static Hedging of Barrier Options in a CEV Model

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Overview

- Barrier options are the most widely used exotic option in the market for foreign exchange.

- In this market, vanilla options are very liquid and hence they can potentially replace the underlying currencies in the replicating portfolio when dynamic replication is feasible.

- Furthermore, *C.Bardos, R.Douady, A.Fursikov* have shown that in all univariate diffusion models, there exists an approximate replicating portfolio of options, which is both static and co-terminal.

- Thus, the payoff to a barrier option of maturity T can be approximately replicated by a portfolio of T maturity puts and calls, which is rebalanced at most once, inside the time interval $(0, T)$.

Marching Forward

- In this paper, we will be working exclusively with forward prices of maturity T. In particular, we will always treat the underlying of the barrier option as a forward FX rate which is continuously monitored.

- We assume no arbitrage and work with a pure discount bond of maturity T as numeraire. We assume no arbitrage and hence the existence of a probability measure $\mathbb{Q}_T$ such that forward prices of all assets are martingales over $t \in [0, T]$.

- We assume that the underlying forward FX rate is a diffusion F under the original probability measure $\mathbb{P}$ and hence it retains sample path continuity and the strong Markov property under $\mathbb{Q}_T$.

Diffusion

- In a univariate diffusion model, the forward FX rate process under the forward measure $\mathbb{Q}_T$ can be modelled as the unique strong solution $\{F_t; t \in [0,T]\}$ to the stochastic differential equation:

$$dF_t = a(F_t, t)dW_t,$$

  for $t \in [0,T]$, and where F_0 is a known positive constant. Here, W is a standard Brownian motion and the function $a(F,t) : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ satisfies growth and Lipschitz restrictions.

- The choice of the state space for the SDE is rectricted by the form of a. For example, when $a(F,t) = a$, a constant, we will let the state space be the whole real line. When $a(F,t) = \sigma F$, for σ constant, we will let the state space be $(0,\infty)$. When $a(F,t) = \delta F^{1+\beta}$, for δ and $\beta \in (-1,0)$ constant, we will let the state space be $[0,\infty)$.

BVP for European Claim

- The forward value function for a European claim with terminal payoff $h(F) : F \in \mathbb{R} \mapsto \mathbb{R}$ is given by the solution $V(F,t), F \in \mathbb{R} \times t \in [0,T] :\mapsto \mathbb{R}$ of a terminal value problem, consisting of the partial differential equation (PDE):

$$\frac{\partial}{\partial t} V(F,t) + \frac{a^2(F,t)}{2} \frac{\partial^2}{\partial F^2} V(F,t) = 0,$$

  defined over the domain $F \in \mathbb{R} \times t \in [0,T]$. The solution is subject to the terminal condition:

$$V(F,T) = h(F),$$

  for $F \in \mathbb{R}$.

- The information in the payoff is propagated backward in time by the PDE (**??**).

- The initial claim value is given by evaluating the value function at the initial forward price F_0 and the initial time:

$$V_0 = V(F_0, 0).$$

Up and In Put

- Let $P(F,t) \equiv V(F,t)$ denote the forward value function when the payoff is that of a European put, i.e. $h(F) = (K-F)^+$.

- Consider an up-and-in put option written on a forward FX rate F with upper barrier $H \in (F_0, \infty)$ and with maturity date $T > 0$.

- The payoff at T is:
$$1(\sup_{t \in [0,T]} F_t > H) \cdot (K - F_T)^+ .$$
Hence the up-and-in put has the same payoff as a standard put at T provided that an upper barrier has been hit or crossed over $(0,T]$.

BVP for UIP

- The forward value of an up-and-in put $U(F,t)$ is the unique solution to a boundary value problem, consisting of the PDE:

$$\frac{\partial}{\partial t}U(F,t)+\frac{a^2(F,t)}{2}\frac{\partial^2}{\partial F^2}U(F,t)=0,$$

  defined on the domain $F \in (-\infty, H) \times t \in [0,T]$.

- The solution is subject to the terminal condition:

$$U(F,T)=0,$$

  for $F \in (-\infty, H)$ and the boundary condition:

$$U(H,t)=P(H,t),$$

  for $t \in [0,T]$.

Existence

- The results of Bardos et. al. imply that for any $\varepsilon > 0$, there exists a payoff function $h^{\varepsilon}(F)$ defined on a closed interval $F \in (\underline{F}, \overline{F})$ such that:

$$\sup_{t \in [0,T]} |V(F,t) - U(F,t)| < \varepsilon,$$

  on the domain where U is defined.

- We conjecture that the reason that they are unable to show that an exact solution exists is due to their requirement that $h(F)$ be defined on a bounded domain.

Construction

- Unfortunately, the proof in Bardos et. al. is not constructive and is of no guidance in finding the payoff function $h(F)$ that would allow one to statically hedge an up-and-in put.

- In our paper, we show that numerical methods do not in general succeed in providing the unknown payoff function $h(F)$.

- However, in the Bachelier and Black models, other authors have shown that the terminal payoff function $h(F)$ is known in closed form.

Back to Bachelier

- In the Bachelier model, the forward price process under the forward measure $\mathbb{Q}_T$ is the solution to:

$$dF_t = a dW_t,$$

for $t \in [0,T]$.

- Here, a is a nonzero constant which practitioners refer to as the normal volatility.

- The payoff whose value matches the put value along the barrier H is just that of a call struck at $2H - K$:

$$h(F) = (F - (2H - K))^+$$

- This is a trivial consequence of the reflection principle first suggested in a Brownian context by Bachelier.

Back to Black

- In the Black model, the risk-neutral forward price dynamics are instead given by:

$$dF_t = \sigma F_t dW_t,$$

for $t \in [0, T]$.

- Here, σ is a nonzero constant, which practitioners refer to as the lognormal volatility.

- The payoff whose value matches the put value along the barrier H is now that of $\frac{K}{H}$ calls struck at H^2/K:

$$h(F) = \frac{K}{H}\left(F - \frac{H^2}{K}\right)^+ .$$

- This result was first suggested by Bowie and Carr in *Risk* 1994. The generalization to constant proportional drift in the underlying is in Carr and Chou. A consequence of adding drift is to require a continuum of options in the hedge.

Independent SV in Bachelier

- So long as the underlying has zero risk-neutral drift, Carr and Lee (2007) show that the same single strike static hedges continue to work when volatility is independently randomized in an unknown way.

- Hence, in the generalization of the Bachelier model to independent and unspecified stochastic volatility, the forward price process under the forward measure $\mathbb{Q}_T$ solves:
$$dF_t = a_t dW_t,$$
for $t \in [0, T]$.

- Here, a is an unknown stochastic process which evolves independently of W. Nonetheless, the payoff whose value matches the put value along the barrier H is still that of a call struck at $2H - K$. As a result, the cost of replicating an up-and-in put is given by the initial market price of this call.

Independent SV in Black

- In the generalization of the Black model to independent and unspecified stochastic volatility, the forward price process under the forward measure $\mathbb{Q}_T$ solves:

$$dF_t = \sigma_t F_t dW_t,$$

  for $t \in [0, T]$.

- Here, σ is an unknown stochastic process which evolves independently of W.

- Nonetheless, the payoff whose value matches the put value along the barrier H is still that of $\frac{K}{H}$ calls struck at H^2/K.

- As a result, the cost of replicating an up-and-in put is given by the initial market price of $\frac{K}{H}$ calls struck at H^2/K.

Zero Drift CEV

- In this paper, we seek to generalize the Bachelier and Black results to a CEV model for the forward price.

- In Cox's CEV model, the forward price process under the forward measure $\mathbb{Q}_T$ is the solution to:
$$dF_t = \delta F_t^{\beta+1} dW_t,$$
for $t \in [0, T]$. Here, δ is a nonzero constant and $\beta \in (-1, 0)$.

- We are able to find an explicit exact solution for the static hedge of an up-and-in put.

- This hedge succeeds even if the parameter δ in the above SDE is replaced by an independent and unknown stochastic process.

Overview of Solution Technique

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- If a static hedge exists, then the forward value of the payoff matches the forward value of a put along the barrier:

$$U(H,t) = P(H,t), t \in [0,T]$$

- The CEV model is time homogeneous so both the unknown claim value and the known put value depend on T and t through time to maturity $\tau \equiv T - t$.

- We (Carson) Laplace transform out τ repalcing PDE's with ODE's which can be solved.

Striking Out?

- Unfortunately, unlike the Bachelier and Black models, the exact static hedge involves standard options arrayed along a continuum of strikes. Furthermore, the payoff function is expressed as an integral over Bessel functions.

- To allow our result to be more easily implemented in practice, we also present an explicit approximation to our static hedge.

- The approximation is determined by requiring that the replication error goes to zero as the time to maturity goes to zero.

Approximation

- The approximate static hedge for an up-and-in put has the same form as the exact hedge in the Black model, i.e. a fixed number of calls at some fixed strike. In fact, we show that as $\beta \uparrow 0$, our approximate CEV model static hedge converges to the exact Black model static hedge. Furthermore, as $\beta \downarrow -1$, our approximate CEV model static hedge converges to the exact Bachelier model static hedge.

- The determination of the hedge weights in our exact static hedge requires copious evaluation of Bessel functions. In contrast, for our approximate hedge, the determination of the call strike and the number of calls held at that strike is exceedingly simple.

K.I.S.S.

- In our approximate static hedge, the strike of the call used in the hedge is just given by:

$$K^* \equiv (2H^{-\beta} - K^{-\beta})^{-\frac{1}{\beta}}.$$

- The number of calls held at this strike is simply:

$$N^* = \left(\frac{K}{K^*}\right)^{\frac{\beta+1}{2}}.$$

- Bye Bye Bessel!

Numerical Tests

- We numerically test the effectiveness of this approximation of our exact static hedge.

- In our preliminary numerical work, we found almost no discrepancy in valuation between the single strike approximate hedge and the exact hedge involving a continuum of strikes.

- Since the CEV model exact static hedge also replicates perfectly when volatility parameter δ is independently randomized into an unkown stochastic process, it follows that the single strike approximation must work well under this (unknown) generalization:

$$dF_t = \delta_t F_t^{\beta+1} dW_t,$$

  for $t \in [0, T]$, and where necessarily $\langle \delta, F \rangle_t = 0$.

Independent SV

- The requirement that the volatility process evolves independently of the Brownian motion driving F is restrictive when $\beta = 0$, i.e. in SV generalizations of the Black model.

- In particular, Carr and Lee (2007) have shown that an unfortunate implication of such models is that the graph of Black implied volatility against $\ln(K/F)$ is even.

- When $\beta \in (-1, 0)$ this graph is not even, When δ is stochastic, Black implied volatility from our independently randomized CEV model need not be downward sloping.

Dependent SV

- We nonetheless compare the value of our approximate static hedge derived assuming zero correlation with the exact static hedge in the SABR model with nonzero correlation.

- We find that the approximate static hedge works across a wide range of correlations.

Summary and Extensions

- When static hedges of barrier options are composed of co-terminal options, they are theoretically attractive for several reasons.

- In particular, when the underlying is a forward price, the same static hedge works across a broad array of models.

- The composition of static hedges of barrier options has been known for the Bachelier and Black models for several years. We generalized these well known results to a CEV process for forward prices.

- The exact result requires integrating Bessel functions but an asymptically exact approximation is computationally trivial

- The approximation also appears to work outside the model but further numerical work is needed.